VL8: Kapitel 3: Real World and Risk Neutral Probabilities, Teil2

In der letzten Vorlesung hatten wir das folgende Theorem bewiesen:

Theorem 3.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model

$$S_k \;=\; S_{k-1} \times \begin{cases} (1+\mathrm{ret_{up}}) & \text{with some probability } \; p \\ (1+\mathrm{ret_{down}}) & \text{with probability } \; 1-p \end{cases}$$

Let $r \geq 0$ be the interest rates and denote by

$$s_k \;=\; e^{-r(t_k - t_0)}\, S_k$$

the discounted price process. Then the following statements hold:

a) Define the risk neutral probability (with $\Delta t = t_k - t_{k-1}$ which we assume to be constant)

$$p_{\mathrm{rn}} \;=\; p_{\text{risk neutral}} \;:=\; \frac{e^{r\Delta t} - 1 \;-\; \mathrm{ret_{down}}}{\mathrm{ret_{up}} \;-\; \mathrm{ret_{down}}}$$

and denote expectations with respect to this probability $p = p_{\mathrm{rn}}$ by $\mathsf{E}_{\mathrm{rn}}[\,\cdot\,]$. Then the discounted price process $\{s_k\}_{k=0}^N$ is a martingale with respect to the risk neutral expectation. That is, the following equation holds for all $k = 0, 1, 2, ..., N-1$:

$$\mathsf{E}_{\mathrm{rn}}\big[\, s_{k+1} \,|\, \{s_j\}_{j=0}^k \,\big] \;=\; s_k$$

b) Let $H = H(S_0, S_1, ..., S_N)$ be the payoff of some option. Then the theoretical fair value of this option can be obtained from the following risk neutral expectation:

$$V_0 \;=\; e^{-r(t_N - t_0)}\, \mathsf{E}_{\mathrm{rn}}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{1}$$

Der wesentliche Punkt war, dass wir eine Wahrscheinlichkeit $p_{\mathrm{rm}} = p_{\text{risk neutral}}$ bestimmen konnten, so dass die Gleichung

$$\mathsf{E}_{\mathrm{rn}}\big[\, s_{k+1} \,|\, \{s_j\}_{j=0}^k \,\big] \;=\; s_k \tag{2}$$

erfüllt ist. Damit war es dann möglich, die delta-Summe in der Formel für den Wert des replizierenden Portfolios zum Verschwinden zu bringen, indem wir da einen Erwartungswert bezüglich der risikoneutralen Wahrscheinlichkeit genommen haben. Auf diese Weise sind wir dann zu der kompakten Pricing-Formel

$$V_0 \;=\; e^{-r(t_N-t_0)}\,\mathsf{E}_{\rm rn}\big[\,H(S_0,S_1,...,S_N)\,\big]$$

gekommen. Bevor wir morgen diese Formel für den Fall einer pfadunabhngigen Funktion $H = H(S_N)$ etwas konkretisieren tun, wollen wir uns an dieser Stelle noch ein bischen mit den bedingten Erwartungswerten $\mathsf{E}\big[\cdots\,\big|\,\{S_j\}_{j=0}^k\big]$ vertraut machen. Dazu machen wir die folgende Übungsaufgabe, wo alle Erwartungswerte mit einer up-Wahrscheinlichkeit von $p_{\rm up} = 1/2$ berechnet werden sollen:

Übungsaufgabe: Wir betrachten ein N-Perioden Binomialmodell mit Preisprozess ($k = 1, 2, ..., N$)

$$S_k \;=\; S_{k-1}\times\begin{cases}(1+q) & \text{mit W'keit } \; p_{\rm up}=1/2\\ (1-q) & \text{mit W'keit } \; p_{\rm down}=1/2\end{cases}$$

mit $q \in (0,1)$ (etwa $N = 250$ und $q = 1\%$) und $S_0 = 100$. Berechnen Sie folgende Erwartungswerte:

a) $\mathsf{E}[\,S_N\,]$

b) $\mathsf{E}\big[\,S_N\,\big|\,\{S_j\}_{j=0}^k\,\big]$

c) $\mathsf{E}\big[\,\frac{1}{N}\sum_{m=1}^N S_m\,\big]$

d) $\mathsf{E}\big[\,\frac{1}{N}\sum_{m=1}^N S_m\,\big|\,\{S_j\}_{j=0}^k\,\big]$

e) $\mathsf{E}[\,\frac{S_0}{S_N}\,]$

f) $\mathsf{E}[\,\frac{S_0}{S_N}\,\big|\,\{S_j\}_{j=0}^k\,\big]$

Bemerkung: Die eigentliche Rechnung ist nicht so schwierig, die Übung besteht mehr darin, sich zu überlegen, ob das Ergebnis eine reine Zahl ist oder ob und welche Buchstaben dann noch im Ergebnis auftauchen.

Lösung: **a)** We have

$$S_N \;=\; S_0\prod_{k=1}^N\big(1+{\rm ret}_k\big)$$

and, since all the returns are independent,

$$\begin{aligned}\mathsf{E}[\,S_N\,] \;&=\; \mathsf{E}\Big[S_0\prod_{k=1}^N\big(1+{\rm ret}_k\big)\Big]\\ &=\; S_0\prod_{k=1}^N\big(1+\mathsf{E}[{\rm ret}_k]\big)\;.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathsf{E}[{\rm ret}_k] \;&=\; +q\times p_{\rm up} \;+\; (-q)\times p_{\rm down}\\ &=\; q/2-q/2 \;=\; 0\end{aligned}$$

Thus,

$$\mathsf{E}[\,S_N\,] \;=\; S_0 \prod_{k=1}^{N} \big(1+0\big) \;=\; S_0\ .$$

b) The notation $\mathsf{E}\big[\cdots \,\big|\, \{S_j\}_{j=0}^k\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned} S_N &= S_0 \prod_{j=1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_0 \prod_{j=1}^{k} \big(1+\mathrm{ret}_j\big) \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \end{aligned}$$

and obtain

$$\begin{aligned} \mathsf{E}\big[\,S_N \,\big|\, \{S_j\}_{j=0}^k\big] &= \mathsf{E}\Big[\,S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \,\big|\, \{S_j\}_{j=0}^k\Big] \\ &= S_k \prod_{j=k+1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_j]\big) \\ &= S_k \prod_{j=k+1}^{N} \big(1+0\big) \;=\; S_k\ . \end{aligned}$$

c) Because of part (a), we have

$$\begin{aligned} \mathsf{E}\Big[\,\frac{1}{N}\sum_{m=1}^{N} S_m\,\Big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}[\,S_m\,] \\ &= \frac{1}{N}\sum_{m=1}^{N} S_0 \;=\; S_0\ . \end{aligned}$$

d) Because of part (b), we obtain

$$\begin{aligned} \mathsf{E}\Big[\,\frac{1}{N}\sum_{m=1}^{N} S_m \,\big|\, \{S_j\}_{j=0}^k\Big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^k\big] \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k} \mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^k\big] \;+\; \sum_{m=k+1}^{N} \mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^k\big]\Big\} \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k} S_m \;+\; \sum_{m=k+1}^{N} S_k\Big\} \\ &= \frac{k}{N}\times\frac{1}{k}\sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N}\times S_k\ . \end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N \;=\; S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} \;=\; \prod_{k=1}^{N} \frac{1}{1+\mathrm{ret}_k}\,.$$

Since all the returns are independent,

$$\begin{aligned}
\mathsf{E}[\,S_0/S_N\,] &= \mathsf{E}\left[\prod_{k=1}^{N} \frac{1}{1+\mathrm{ret}_k}\right] \\
&= \prod_{k=1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_k}\right] \\
&= \prod_{k=1}^{N} \left\{\frac{1}{1+q}\times\frac{1}{2}+\frac{1}{1-q}\times\frac{1}{2}\right\} \\
&= \prod_{k=1}^{N} \left\{\frac{1}{1-q^2}\right\} \\
&= \frac{1}{(1-q^2)^N}\,.
\end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots\big|\,\{S_j\}_{j=0}^{k}\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write as in part (b)

$$S_0 \,/\, S_N \;=\; S_0 \,/\, \Big\{S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big)\Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}[\,S_0/S_N \,\big|\, \{S_j\}_{j=0}^{k}\,] &= S_0/S_k\; \mathsf{E}\left[\prod_{m=k+1}^{N} \frac{1}{1+\mathrm{ret}_m}\,\Big|\,\{S_j\}_{j=0}^{k}\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_m}\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1+q}\times\frac{1}{2}+\frac{1}{1-q}\times\frac{1}{2}\right\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1-q^2}\right\} \\
&= S_0/S_k\; \frac{1}{(1-q^2)^{N-k}}\,. \qquad \blacksquare
\end{aligned}$$

Bevor wir morgen die Pricing-Formel für den Fall von pfadunabhängigen Optionen konkretisieren tun, wollen wir noch ein paar kleinere Sachen festhalten:

Corollary 3.1: Consider a price process $S_k = S(t_k)$ given by a Binomial model. Let $r \geq 0$ be the interest rates and denote by

$$v_k \;=\; e^{-r(t_k-t_0)}\, V_k \tag{3}$$

the discounted time t_k portfolio value of the replicating portfolio V_k. Then the process $\{v_k\}_{k=0}^{N}$ is a martingale with respect to expectations with the risk neutral probability (1). That is, the following equation holds:

$$\mathsf{E}_{\mathrm{rn}}\big[\,v_{k+1}\,|\,\{S_j\}_{j=0}^{k}\,\big] \;=\; v_k \tag{4}$$

for all $k = 0, 1, 2, ..., N-1$.

Proof: According to part (b) of Theorem 1.1, our standard formula with nonzero interest rates, we have

$$v_k \;=\; v_0 \;+\; \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1})$$

from which we get

$$v_{k+1} \;=\; v_k + \delta_k(s_{k+1} - s_k) \tag{5}$$

Thus, since $v_k = v_k(S_0, .., S_k)$ and $\delta_k = \delta_k(S_0, .., S_k)$ do not depend on S_{k+1}

$$\begin{aligned}
\mathsf{E}_{\rm rn}\big[\, v_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] &= \mathsf{E}_{\rm rn}\big[\, v_k + \delta_k(s_{k+1} - s_k) \,|\, \{S_j\}_{j=0}^k \,\big] \\
&= v_k + \delta_k \times \mathsf{E}_{\rm rn}\big[\, s_{k+1} - s_k \,|\, \{S_j\}_{j=0}^k \,\big] \\
&= v_k + \delta_k \times \Big(\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;-\; s_k \,\Big) \\
&= v_k
\end{aligned}$$

where we used again the martingale property $\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] = s_k$ in the last line. ■

Remark: Equation (4) of the Corollary 3.1 is actually equivalent to the recursion relation of Theorem 2.1, that was the following equation

$$v_k \;=\; w_{\rm up}\, v_{k+1}^{\rm up} \;+\; w_{\rm down}\, v_{k+1}^{\rm down}$$

with weights $w_{\rm up/down}$ given by

$$\begin{aligned}
w_{\rm up} &= \frac{(e^{r\Delta t} - 1) \;-\; {\rm ret}_{\rm down}}{{\rm ret}_{\rm up} \;-\; {\rm ret}_{\rm down}} \\
w_{\rm down} \;=\; 1 - w_{\rm up} &= \frac{{\rm ret}_{\rm up} \;-\; (e^{r\Delta t} - 1)}{{\rm ret}_{\rm up} \;-\; {\rm ret}_{\rm down}}
\end{aligned}$$

Apparently, $w_{\rm up}$ is identical to the risk neutral probability $p_{\rm rn}$,

$$w_{\rm up} \;=\; p_{\rm rn}$$

such that the recursion of Theorem 2.1 can also be written as

$$\begin{aligned}
v_k &= w_{\rm up}\, v_{k+1}^{\rm up} \;+\; w_{\rm down}\, v_{k+1}^{\rm down} \\
&= p_{\rm rn}\, v_{k+1}^{\rm up} \;+\; (1 - p_{\rm rn})\, v_{k+1}^{\rm down} \\
&= p_{\rm rn}\, v_{k+1}\big(S_0, \cdots, S_k, S_k(1 + {\rm ret}_{\rm up})\big) \;+\; (1 - p_{\rm rn})\, v_{k+1}\big(S_0, \cdots, S_k, S_k(1 + {\rm ret}_{\rm down})\big) \\
&= \mathsf{E}_{\rm rn}\Big[\, v_{k+1}\big(S_0, \cdots, S_k, S_k(1 + {\rm ret}_{k+1})\big) \;\big|\; \{S_j\}_{j=1}^k \,\Big] \\
&= \mathsf{E}_{\rm rn}\big[\, v_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;. \qquad ■
\end{aligned}$$