VL5: Kapitel 2: Das Binomialmodell, Teil2

Gestern hatten wir das Binmialmodell als ein zeitdiskretes Assetpreismodell definiert, was von einem Zeitpunkt t_{k-1} zum nächsten Zeitpunkt t_k jeweils immer nur 2 Einstellungmöglichkeiten zulässt, das war die

Definition 2.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$S_k \;=\; S_{k-1}(\,1 \,+\, \mathrm{ret}_k\,) \qquad \text{with} \quad \mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\} \tag{1}$$

for all k, then we say that S is given by the Binomial model.

Dann hatten wir das folgende Theorem hingeschrieben, wo wir uns jetzt noch den Beweis dazu anschauen wollen:

Theorem 2.1: Let S be some tradable asset whose price process is given by the Binomial model (1). Let $r \geq 0$ denote some constant interest rate. Then every option payoff $H = H(S_0, ..., S_N)$ can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \delta_k(S_0, \cdots, S_k) \;=\; \frac{V_{k+1}^{\mathrm{up}} \,-\, V_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} \,-\, S_{k+1}^{\mathrm{down}}} \tag{2}$$

with the abbreviations

$$\begin{aligned} S_{k+1}^{\mathrm{up/down}} \;&:=\; S_k\,(\,1 \,+\, \mathrm{ret}_{\mathrm{up/down}}\,) \\ V_{k+1}^{\mathrm{up/down}} \;&:=\; V_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\mathrm{up/down}}\,\big) \end{aligned}$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae: Suppose continuous interest rates r. Then

$$V_k \;=\; e^{rt_k}\, v_k$$

with discounted portfolio values v_k given inductively by

$$v_k \;=\; w_{\mathrm{up}}\, v_{k+1}^{\mathrm{up}} \,+\, w_{\mathrm{down}}\, v_{k+1}^{\mathrm{down}} \tag{3}$$

$$v_N \;:=\; e^{-rt_N}\, H(S_0, \cdots, S_N)$$

with weights

$$w_{\text{up}} \;=\; \frac{(e^{r\Delta t}-1)\;-\;\text{ret}_{\text{down}}}{\text{ret}_{\text{up}}\;-\;\text{ret}_{\text{down}}} \tag{4}$$

$$w_{\text{down}} \;=\; 1-w_{\text{up}} \;=\; \frac{\text{ret}_{\text{up}}\;-\;(e^{r\Delta t}-1)}{\text{ret}_{\text{up}}\;-\;\text{ret}_{\text{down}}} \tag{5}$$

and $\Delta t := t_k - t_{k-1}$ (which we assume to be constant, independent of k).

Proof of Theorem 2.1: For nonzero interest rates we have

$$v_N \;=\; v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$

which is equivalent to

$$v_{k+1} \;=\; v_k + \delta_k(s_{k+1} - s_k) \qquad \forall\; k = 0, 1, ..., N-1 \tag{6}$$

We have

$$v_{k+1} \;=\; v_{k+1}(S_0, \cdots, S_k, S_{k+1}) \;=\; v_{k+1}\big(\, S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\text{ret}_k)\,\big)$$

and the return ret_k can be an up-move or a down-move in which case we get

$$\begin{aligned} v_{k+1}^{\text{up}} \;&=\; v_{k+1}\big(\, S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\text{up}}\,\big) \;=\; v_{k+1}\big(\, S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\text{ret}_{\text{up}})\,\big) \\ v_{k+1}^{\text{down}} \;&=\; v_{k+1}\big(\, S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\text{down}}\,\big) \;=\; v_{k+1}\big(\, S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\text{ret}_{\text{down}})\,\big) \end{aligned}$$

From (6), we have

$$\begin{aligned} v_{k+1}^{\text{up}} \;&=\; v_k + \delta_k(s_{k+1}^{\text{up}} - s_k) \\ v_{k+1}^{\text{down}} \;&=\; v_k + \delta_k(s_{k+1}^{\text{down}} - s_k) \end{aligned}$$

Thus,

$$v_{k+1}^{\text{up}} \;-\; v_{k+1}^{\text{down}} \;=\; \delta_k(s_{k+1}^{\text{up}} \;-\; s_{k+1}^{\text{down}})$$

or

$$\delta_k \;=\; \frac{v_{k+1}^{\text{up}} \;-\; v_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} \;-\; s_{k+1}^{\text{down}}} \;=\; \frac{e^{-rt_{k+1}}(V_{k+1}^{\text{up}} \;-\; V_{k+1}^{\text{down}})}{e^{-rt_{k+1}}(S_{k+1}^{\text{up}} \;-\; S_{k+1}^{\text{down}})} \;=\; \frac{V_{k+1}^{\text{up}} \;-\; V_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} \;-\; S_{k+1}^{\text{down}}}$$

Solving (6) for v_k,

$$\begin{aligned} v_k \;&=\; v_{k+1} \;-\; \delta_k(s_{k+1} - s_k) \\ &=\; v_{k+1}^{\text{up}} \;-\; \delta_k(s_{k+1}^{\text{up}} - s_k) \\ &=\; v_{k+1}^{\text{down}} \;-\; \delta_k(s_{k+1}^{\text{down}} - s_k) \end{aligned}$$

Let's take the up-equation and substitute the value for δ_k (we also could use the down-equation, we would end up with the same result),

$$\begin{aligned}
v_k &= v_{k+1}^{\text{up}} - \delta_k(s_{k+1}^{\text{up}} - s_k) \\
&= v_{k+1}^{\text{up}} - \tfrac{v_{k+1}^{\text{up}} - v_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\,(s_{k+1}^{\text{up}} - s_k) \\
&= \tfrac{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\, v_{k+1}^{\text{up}} - (v_{k+1}^{\text{up}} - v_{k+1}^{\text{down}})\,\tfrac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} \\
&= \tfrac{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}} - s_{k+1}^{\text{up}} + s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\, v_{k+1}^{\text{up}} + \tfrac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\, v_{k+1}^{\text{down}} \\
&= \tfrac{s_k - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\, v_{k+1}^{\text{up}} + \tfrac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}\, v_{k+1}^{\text{down}} \\
&=: w_{\text{up}}\, v_{k+1}^{\text{up}} + w_{\text{down}}\, v_{k+1}^{\text{down}}
\end{aligned}$$

with weights w_{up} and w_{down} which apparently add up to 1 and

$$w_{\text{up}} = \frac{s_k - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} = \frac{e^{-rt_k} S_k - e^{-rt_{k+1}} S_{k+1}^{\text{down}}}{e^{-rt_{k+1}}(S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}})} = \frac{e^{r(t_{k+1}-t_k)} S_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}}$$

or

$$\begin{aligned}
w_{\text{up}} &= \frac{e^{r\Delta t} S_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} \\
&= \frac{e^{r\Delta t} S_k - S_k(1 + \text{ret}_{\text{down}})}{S_k(1 + \text{ret}_{\text{up}}) - S_k(1 + \text{ret}_{\text{down}})} \\
&= \frac{e^{r\Delta t} - 1 - \text{ret}_{\text{down}}}{(1 + \text{ret}_{\text{up}}) - (1 + \text{ret}_{\text{down}})} \\
&= \frac{e^{r\Delta t} - 1 - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}
\end{aligned}$$

and the theorem is proven. ■

Remarks: 1) If H is a path-independent or non-exotic option which depends only on the underlying price at maturity,

$$H = H(S_N)$$

then the δ_k and the value of the replicating portfolio V_k at t_k depend only on the asset price S_k and do not depend on earlier prices $S_{k-1}, S_{k-2}, ..., S_0$. That is,

$$\begin{aligned}
V_k &= V_k(S_k) \\
\delta_k &= \delta_k(S_k)
\end{aligned}$$

2) Assume zero interest rates such that $e^{r\Delta t} = 1$ and $v_k = V_k$. Then (3) becomes

$$V_k = w_{\text{up}}\, V_{k+1}^{\text{up}} + w_{\text{down}}\, V_{k+1}^{\text{down}}$$

with weights

$$\begin{aligned} w_{\text{up}} &= \frac{e^{r\Delta t} - 1 \; - \; \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \; - \; \text{ret}_{\text{down}}} \;=\; \frac{- \, \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \; - \; \text{ret}_{\text{down}}} \\ w_{\text{down}} &= \frac{+ \, \text{ret}_{\text{up}}}{\text{ret}_{\text{up}} \; - \; \text{ret}_{\text{down}}} \end{aligned}$$

If we further assume a 'symmetric' Binomial model with (say, $q = 1\%$ or $q = 5\%$)

$$\begin{aligned} \text{ret}_{\text{up}} &= +q \\ \text{ret}_{\text{down}} &= -q \end{aligned}$$

the weights simplify to

$$w_{\text{up}} = \frac{-(-q)}{2q} = \frac{1}{2} = w_{\text{down}}$$

and we arrive at the simple recursion formula

$$V_k = \frac{V_{k+1}^{\text{up}} \; + \; V_{k+1}^{\text{down}}}{2} \tag{7}$$

Schauen wir uns jetzt ein konkretes Beispiel dazu an, das machen wir auf einem Excelsheet → nächste Woche.