VL4: Kapitel 2: Das Binomialmodell, Teil1

Let S be some tradable asset with prices

$$S_k \;=\; S(t_k), \quad k = 0, 1, 2, ... \tag{1}$$

and let

$$H \;=\; H(S_0, S_1, ..., S_{N-1}, S_N) \tag{2}$$

be some option payoff with start date t_0 and end date or maturity t_N. We want to replicate the option payoff (2) with a suitable trading strategy in the underlying S. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding δ_k assets at the end of day t_k generates the amount

$$V_N \;=\; V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{3}$$

Each δ_k will be determined on the end of trading day t_k. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, δ_k can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k \;=\; \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \tag{4}$$

Definition 2.1: We say that an option payoff (2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of δ_k of the form (4) and some initial amount V_0 such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_0 \;+\; \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{5}$$

The **initial amount $\mathbf{V_0}$** which is needed to set up the replicating strategy is called the theoretical fair value of H or **the price of the option H.** The process of replicating an option payoff H through formula (5), that is, through a trading strategy which holds δ_j pieces of the underlying S at the end of day t_j, is called **hedging**.

Now let us consider the question to what extent replication of options is possible. Equation (5) can be rewritten as

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$
\begin{aligned}
H(S_0, S_1, ..., S_{N-1}, S_N) - \delta_{N-1}\, S_N &= V_{N-1} - \delta_{N-1}\, S_{N-1} \\
&= \text{some function of } \; S_0, S_1, ..., S_{N-1} \qquad (6)
\end{aligned}
$$

That is, the right hand side of (6) is independent of S_N. Let us introduce the return of the asset S from t_{k-1} to t_k,

$$
\mathrm{ret}_k \;:=\; \frac{S_k - S_{k-1}}{S_{k-1}} \;=\; \frac{S_k}{S_{k-1}} \;-\; 1 \qquad (7)
$$

such that

$$
S_k \;=\; S_{k-1}(1 + \mathrm{ret}_k) \qquad (8)
$$

Then equation (6) can be rewritten as

$$
H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \mathrm{ret}_N)\big) - \delta_{N-1}\, S_{N-1}(1 + \mathrm{ret}_N) \;=\; \text{const} \qquad (9)
$$

where the const in the equation above means that the left hand side of (9) has to be the same for all possible choices of ret_N. Since there is only 1 free parameter in (9), namely δ_{N-1}, we can only allow for 2 possible choices for ret_N, say,

$$
\mathrm{ret}_N \;\in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\} \qquad (10)
$$

and in that case we have to have

$$
\begin{aligned}
&H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \mathrm{ret}_{\mathrm{up}})\big) - \delta_{N-1}\, S_{N-1}(1 + \mathrm{ret}_{\mathrm{up}}) \;= \\
&H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \mathrm{ret}_{\mathrm{down}})\big) - \delta_{N-1}\, S_{N-1}(1 + \mathrm{ret}_{\mathrm{down}})
\end{aligned}
$$

which determines δ_{N-1} to

$$
\delta_{N-1} \;=\; \frac{H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \mathrm{ret}_{\mathrm{up}})\big) \;-\; H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \mathrm{ret}_{\mathrm{down}})\big)}{S_{N-1}(1 + \mathrm{ret}_{\mathrm{up}}) \;-\; S_{N-1}(1 + \mathrm{ret}_{\mathrm{down}})} \qquad (11)
$$

Thus, if we allow for 2 possible choices of returns only as in (10), replication of option payoffs seem to be possible. This leads to the following

Definition 2.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$
S_k \;=\; S_{k-1}(\,1 \;+\; \mathrm{ret}_k\,) \qquad \text{with} \quad \mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\} \qquad (12)
$$

for all k, then we say that S is given by the Binomial model.

Remark: Observe that in Definition 2.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability p_{up} such that an up-return $\mathrm{ret}_{\mathrm{up}}$ will occur and a probability $p_{\mathrm{down}} = 1 - p_{\mathrm{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value V_0 are actually independent of such probabilities.

Now we are in a position to formulate the following important

Theorem 2.1: Let S be some tradable asset whose price process is given by the Binomial model (12). Let $r \geq 0$ denote some constant interest rate. Then every option payoff

$$H \;=\; H(S_0, ..., S_N)$$

can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \delta_k(S_0, \cdots, S_k) \;=\; \frac{V_{k+1}^{\text{up}} \,-\, V_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} \,-\, S_{k+1}^{\text{down}}} \tag{13}$$

with the abbreviations

$$\begin{aligned} S_{k+1}^{\text{up/down}} &:= S_k\,(1 \,+\, \text{ret}_{\text{up/down}}) \\ V_{k+1}^{\text{up/down}} &:= V_{k+1}\big(S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\text{up/down}}\big) \end{aligned}$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae: Suppose continuous interest rates r. Then

$$V_k \;=\; e^{rt_k}\,v_k$$

with discounted portfolio values v_k given recursively by

$$v_k \;=\; w_{\text{up}}\,v_{k+1}^{\text{up}} \,+\, w_{\text{down}}\,v_{k+1}^{\text{down}} \tag{14}$$

and the recursion starts at $k = N$ with discounted portfolio values

$$v_N \;:=\; e^{-rt_N}\,H(S_0, \cdots, S_N)$$

The weights w_{up} and w_{down} are given by

$$w_{\text{up}} \;=\; \frac{(e^{r\Delta t} - 1) \,-\, \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \,-\, \text{ret}_{\text{down}}} \tag{15}$$

$$w_{\text{down}} \;=\; 1 - w_{\text{up}} \;=\; \frac{\text{ret}_{\text{up}} \,-\, (e^{r\Delta t} - 1)}{\text{ret}_{\text{up}} \,-\, \text{ret}_{\text{down}}} \tag{16}$$

and $\Delta t := t_k - t_{k-1}$ (which we assume to be constant, independent of k).

Remark: The quantity $e^{r\Delta t} - 1$ is actually the interest rate paid per period when going from t_{k-1} to t_k, so in the context of the Binomial model it is probably more natural to use diskrete Verzinsung and then we would say that we have a discretely paid interest rate $\tilde{r}$ per period with $\tilde{r}$ be given by

$$\tilde{r} \;=\; e^{r\Delta t} - 1$$

Then the formula for the weights $w_{\text{up/down}}$ is simply

$$\begin{aligned} w_{\text{up}} &= \frac{\tilde{r} - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\ w_{\text{down}} &= \frac{\text{ret}_{\text{up}} - \tilde{r}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \end{aligned}$$

Proof of Theorem 2.1: ..können wir uns morgen anschauen.

Morgen werden wir dann auch ein konkretes Beispiel durchrechnen; wenn Sie das gesehen haben, haben Sie dann auch ein ganz gutes Gefühl für die Sache, so jetzt vielleicht noch nicht so ganz.