Kapitel 9: Das risikoneutrale Pricing-Maß für das Black-Scholes Modell und Monte Carlo Evaluation, Teil2

Letzte Woche hatten wir das folgende Theorem bewiesen:

Theorem 9.1: Let $s_{t_k} = s(x_{t_k}, t_k)$ be a discounted geometric Brownian motion given by

$$s_{t_k} \;=\; e^{-rt_k}\, S_{t_k} \;=\; e^{-rt_k}\, S_0\, e^{(\mu-\frac{\sigma^2}{2})t_k+\sigma x_{t_k}} \;=\; S_0\, e^{\sigma x_{t_k}+(\mu-r-\sigma^2/2)t_k} \;=:\; s(x_{t_k}, t_k) \tag{1}$$

Define the kernels $\tilde{p}_t(x,y) = \tilde{p}_t^{\mu,r,\sigma}(x,y)$ by

$$\tilde{p}_t(x,y) \;:=\; \tfrac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y-\frac{\mu-r}{\sigma}t)^2}{2t}} \tag{2}$$

Then:

a) The kernels $\tilde{p}$ satisfy

$$\int_{\mathbb{R}} \tilde{p}_s(x,y)\, \tilde{p}_t(y,z)\, dy \;=\; \tilde{p}_{s+t}(x,z) \tag{3}$$

and $\int_{\mathbb{R}} \tilde{p}_s(x,y)\, dy = 1$ for all x. That is, the measure

$$d\tilde{W}(\{x_t\}_{0<t\le T}) \;:=\; \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} \tilde{p}_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{4}$$

is well defined.

b) The price process (1) is a martingale with respect to $d\tilde{W}$. That is,

$$\begin{aligned} \mathsf{E}_{\tilde{W}}\big[s(x_{t'}, t') \,|\, x_t\big] \;&:=\; \int s(x_{t'}, t')\, d\tilde{W}(\{x_s\}_{t<s\le T}) \\ &=\; s(x_t, t) \quad \forall\, t < t' \end{aligned} \tag{5}$$

The measure $d\tilde{W}$ is called an equivalent martingale measure (with respect to the price process s_t).

c) Wir können den Preis einer beliebigen pfadabhängigen oder pfadunabhängigen Option mit Payoff $H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)})$ schreiben als

$$V_0 \;=\; e^{-rT}\, \mathsf{E}_{\tilde{W}}\big[H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)})\big] \tag{6}$$

Dabei bezeichnet

$$S_t^{(\mu)} \;:=\; S_0\, e^{\sigma x_t + (\mu-\frac{\sigma^2}{2})t}$$

den real world Preisprozess.

Wir können also, ähnlich wie im Binomialmodell, auch im Black-Scholes Modell die Preise von beliebigen pfadabhängigen oder pfadunabhängigen Optionen einfach als Erwartungswert vom Payoff schreiben. Allerdings taucht in der Formel (6) das μ auf, der real world Drift-Parameter, den wir ja nicht kennen, den wir statistisch nicht verlässlich schätzen können (im Gegensatz zum σ, das ist ein deutlich stabilerer Parameter). Glücklicherweise stellt sich nun heraus, dass der Ausdruck auf der rechten Seite von (6) tatsächlich unabhängig ist von μ, das wollen wir uns jetzt noch genauer anschauen:

In chapter 5 where we approximated the Black-Scholes model with a suitable Binomial model, we were able to prove the following pricing formula for some non path dependent option with payoff $H = H(S_T)$, see Theorem 5.2:

$$V_0^{\rm BS} \;=\; e^{-rT}\int_{\mathbb{R}} H\bigl(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}\bigr)e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \tag{7}$$

Let us rederive (7) by using the equivalent martingale measure. Suppose we have a european option with discounted payoff $h(S_T) = e^{-rT}H(S_T)$ and that the dynamics of S is given by the Black-Scholes model with real world drift μ and volatility σ,

$$S_T \;=\; S_0\, e^{\sigma x_T+(\mu-\frac{\sigma^2}{2})T} \tag{8}$$

Observe that the drift parameter μ does not show up in the pricing formula (7). This was actually a quite fundamental result of chapter 5. Here we will come up with the same conclusion:

Let δ be the replicating strategy. Then

$$v_T \;=\; h(S_T) \;=\; v_0 + \sum_{k=1}^{N_T}\delta_{t_{k-1}}(s_{t_k}-s_{t_{k-1}}) \tag{9}$$

We fix some $t = t_k$ and take the expectation with respect to $d\tilde{W}(\{x_s\}_{t<s\le T})$. This gives

$$\mathsf{E}_{\tilde W}\bigl[h(S_T)|x_t\bigr] \;=\; v_0 + \sum_{j=1}^{k}\delta_{t_{j-1}}(s_{t_j}-s_{t_{j-1}}) \;=\; v_{t_k} \;=\; v_t \;=\; e^{-rt}V_t \tag{10}$$

and we get

$$\begin{aligned}
V_t \;&=\; e^{rt}\,\mathsf{E}_{\tilde W}\bigl[h(S_T)|x_t\bigr] \\
&=\; e^{-r(T-t)}\int_{\mathbb{R}} H\bigl(S_0\, e^{\sigma x_T+(\mu-\frac{\sigma^2}{2})T}\bigr)\,\tilde p_{T-t}(x_t,x_T)\,dx_T \\
&=\; e^{-r(T-t)}\int_{\mathbb{R}} H\bigl(S_0\, e^{\sigma x_T+(\mu-\frac{\sigma^2}{2})T}\bigr)\, e^{-\frac{\left(x_t-x_T-\frac{\mu-r}{\sigma}(T-t)\right)^2}{2(T-t)}}\;\frac{dx_T}{\sqrt{2\pi(T-t)}} \\
&=\; e^{-r(T-t)}\int_{\mathbb{R}} H\bigl(S_t\, e^{\sigma(x_T-x_t)+(\mu-\frac{\sigma^2}{2})(T-t)}\bigr)\, e^{-\frac{\left(x_t-x_T-\frac{\mu-r}{\sigma}(T-t)\right)^2}{2(T-t)}}\;\frac{dx_T}{\sqrt{2\pi(T-t)}} \\
&=\; e^{-r(T-t)}\int_{\mathbb{R}} H\bigl(S_t\, e^{\sigma(x_T-x_t)+(\mu-r-\frac{\sigma^2}{2})(T-t)}\, e^{r(T-t)}\bigr)\, e^{-\frac{\left(x_T-x_t+\frac{\mu-r}{\sigma}(T-t)\right)^2}{2(T-t)}}\;\frac{dx_T}{\sqrt{2\pi(T-t)}}
\end{aligned}$$

We substitute $x_T \to y$ according to

$$\begin{aligned}
y \;&:=\; x_T - x_t + \tfrac{\mu-r}{\sigma}(T-t) \\
dy \;&=\; dx_T
\end{aligned}$$

and obtain

$$
\begin{aligned}
V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H\Big(S_t\, e^{\sigma(x_T-x_t)\,+\,(\mu-r)(T-t)}\, e^{(r\,-\,\frac{\sigma^2}{2})(T-t)}\Big)\, e^{-\frac{\left(x_T-x_t\,+\,\frac{\mu-r}{\sigma}(T-t)\right)^2}{2(T-t)}}\, \frac{dx_T}{\sqrt{2\pi(T-t)}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}} H\Big(S_t\, e^{\sigma y}\, e^{(r-\frac{\sigma^2}{2})(T-t)}\Big)\, e^{-\frac{y^2}{2(T-t)}}\, \frac{dy}{\sqrt{2\pi(T-t)}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}} H\Big(S_t\, e^{\sigma\sqrt{T-t}\,x\,+\,(r-\frac{\sigma^2}{2})(T-t)}\Big)\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \qquad (11)
\end{aligned}
$$

and, for $t = 0$, this coincides with (7). More generally, there holds the following

Theorem 9.2: Let $\{x_t\}_{0<t\leq T}$ be a Brownian motion and let's use the following notation for the real world and risk neutral price paths:

$$S_t^{(\mu)} \;:=\; S_0\, e^{\sigma x_t\,+\,(\mu-\frac{\sigma^2}{2})t} \qquad (12)$$

$$S_t^{(r)} \;:=\; S_0\, e^{\sigma x_t\,+\,(r-\frac{\sigma^2}{2})t} \qquad (13)$$

Let dW be the Wiener measure and let $d\tilde{W}$ be the equivalent martingale measure of Theorem 9.1. Then the following equality holds for arbitrary path dependent or path independent payoffs $H = H(S_{t_1}, ..., S_{t_m})$:

$$\mathsf{E}_{\tilde{W}}\big[H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)})\big] \;=\; \mathsf{E}_{W}\big[H(S_{t_1}^{(r)}, ..., S_{t_m}^{(r)})\big] \qquad (14)$$

In particular, the theoretical fair value

$$
\begin{aligned}
V_0 &\overset{\text{Theorem 9.1}}{=} e^{-rT}\mathsf{E}_{\tilde{W}}\big[H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)})\big] \\
&\overset{(14)}{=} e^{-rT}\mathsf{E}_{W}\big[H(S_{t_1}^{(r)}, ..., S_{t_m}^{(r)})\big] \qquad (15)
\end{aligned}
$$

does not depend on the (usually not predictable) drift parameter μ, but depends only on the volatility parameter σ and the interest rate level r.

Proof: From the definition of $\tilde{p}$ we have ($t_0 := 0$)

$$\mathsf{E}_{\tilde{W}}\big[H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)})\big] \;=\; \int_{\mathbb{R}^m} H(S_{t_1}^{(\mu)}, ..., S_{t_m}^{(\mu)}) \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\, e^{-\frac{\left[x_{t_{j-1}}+\frac{\mu-r}{\sigma}t_{j-1}\,-\,(x_{t_j}+\frac{\mu-r}{\sigma}t_j)\right]^2}{2(t_j-t_{j-1})}}\, dx_{t_j}$$

Because of

$$
\begin{aligned}
S_t^{(\mu)} \;=\; S_t^{(\mu)}(x_t) &= S_0\, e^{\sigma x_t\,+\,(\mu-\frac{\sigma^2}{2})t} \\
&= S_0\, e^{\sigma\left(x_t\,+\,\frac{\mu-r}{\sigma}t\right)\,+\,(r-\frac{\sigma^2}{2})t} \\
&= S_t^{(r)}(x_t \,+\, \tfrac{\mu-r}{\sigma}t) \qquad (16)
\end{aligned}
$$

the statement follows from the substitution of variables $y_{t_j} = x_{t_j} + \frac{\mu-r}{\sigma}t_j$. ■

Monte Carlo Evaluation:

We have the following basic pricing formula:

$$V_0 \;=\; e^{-rT}\,\mathsf{E}_W\big[H(S^{(r)}_{t_1},...,S^{(r)}_{t_m})\big] \tag{17}$$

Monte Carlo evaluation goes as follows (at this place we neglect any theoretical motivations or derivations):

Choose a number N of Monte Carlo paths (typically $N = 100'000$ or $N = 1'000'000$) and simulate N price paths for the risk neutral price process through ($t_k = k\Delta t$)

$$S^{(r)}_{t_k} \;=\; S^{(r)}_{t_{k-1}}\big(1 + r\Delta t + \sigma\sqrt{\Delta t}\,\phi_k\big) \tag{18}$$

with the ϕ_k being standard normal independent random numbers. Then the following approximation holds:

$$\mathsf{E}_W\big[H(S^{(r)}_{t_1},...,S^{(r)}_{t_m})\big] \;\approx\; \frac{1}{\text{number of paths}}\sum_{\text{paths}} H(\text{path}) \tag{19}$$

and the relative error is of size $O\big(\frac{1}{\sqrt{N}}\big)$. One may also express this through the following formula:

$$\frac{1}{\text{number of paths}}\sum_{\text{paths}} H(\text{path}) \;=\; \mathsf{E}_W\big[\,H(\{S^{(r)}_t\}_{0\leq t\leq T})\,\big]\times\Big(\;1\;+\;O\big(\tfrac{1}{\sqrt{N}}\big)\;\Big)$$

That is, the Monte Carlo error is of size $O(1/\sqrt{\text{number of paths}})$. In order to obtain a relative precision of $0.1\% = 1/1000$ (which would mean, say, exact price 12.74 versus MC price of 12.72 or 12.77 or so), a number of $N = 1'000'000$ paths has to be used.

Machen wir dazu noch eine Excel/VBA-Simulation: Wir wollen die Black-Scholes Formeln für Standard Call- und Put-Optionen mit einer Monte Carlo Simulation verifizieren, das schauen wir uns morgen noch an.