Kapitel 9: Das risikoneutrale Pricing-Maß für das Black-Scholes Modell und Monte Carlo Evaluation

The discounted portfolio value of a selffinancing strategy in discrete time was given by

$$v_{t_k} \;=\; v_0 + \sum_{j=1}^{k} \delta_{t_{j-1}}(s_{t_j} - s_{t_{j-1}}) \tag{1}$$

where $v_{t_k} = e^{-rt_k} V_{t_k}$ is the discounted portfolio value at time $t_k = k\Delta t$, $k = 0, 1, ..., N_T = T/\Delta t$ and $s_{t_k} = e^{-rt_k} S_{t_k}$ denotes the discounted asset price. Suppose that the price process S_{t_k} is given by a geometric Brownian motion such that

$$s_{t_k} \;=\; e^{-rt_k}\, S_0\, e^{(\mu-\frac{\sigma^2}{2})t_k + \sigma x_{t_k}} \;=\; S_0\, e^{\sigma x_{t_k} + (\mu - r - \sigma^2/2)t_k} \;=:\; s(x_{t_k}, t_k) \tag{2}$$

Suppose further that we have some european option with payoff H or discounted payoff $h = e^{-rT} H$ where $h = h(S_{t_N})$ or more generally $h = h(S_{t_0}, ..., S_{t_N})$, $N = N_T$, and suppose that there is a replicating strategy

$$\delta_{t_k} \;=\; \delta_{t_k}(S_{t_1}, ..., S_{t_k}) \tag{3}$$

such that

$$h \;=\; v_{t_N} \;=\; v_0 + \sum_{k=1}^{N} \delta_{t_{k-1}}(s_{t_k} - s_{t_{k-1}}) \tag{4}$$

Since $v_0 = V_0$, the price of the option, is a number, we can write

$$v_0 \;=\; \mathsf{E}[\, v_0 \,] \;=\; \mathsf{E}[\, h \,] - \sum_{k=1}^{N} \mathsf{E}\big[\, \delta_{t_{k-1}}(s_{t_k} - s_{t_{k-1}}) \,\big] \tag{5}$$

where the expectation in (5) can be choosen arbitrarily. Let us consider (5) for the Wiener measure,

$$dW(\{x_t\}_{0<t\leq T}) \;=\; \lim_{\Delta t \to 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{6}$$

Because of Theorem 4.1 we have

$$\begin{aligned} &\mathsf{E}_W\big[\delta_{k-1}(s_k - s_{k-1})\big] \qquad (7) \\ &\quad= \int_{\mathbb{R}^k} \delta_{k-1}(x_{t_1}, ..., x_{t_{k-1}})\big(s(x_{t_k}, t_k) - s(x_{t_{k-1}}, t_{k-1})\big) \prod_{j=1}^{k} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j})\, dx_{t_j} \end{aligned}$$

The integration variable x_{t_k} shows up only at one place such that we can write

$$
\begin{aligned}
&\mathsf{E}_W\left[\delta_{k-1}(s_k - s_{k-1})\right] \\
&\quad = \int_{\mathbb{R}^{k-1}} \delta_{k-1}(x_{t_1}, ..., x_{t_{k-1}}) \Big(\textstyle\int_{\mathbb{R}} s(x_{t_k}, t_k)\, p_{t_k - t_{k-1}}(x_{t_{k-1}}, x_{t_k})\, dx_{t_k} - s(x_{t_{k-1}}, t_{k-1}) \Big) \times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \prod_{j=1}^{k-1} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j})\, dx_{t_j} \\
&\quad = \int_{\mathbb{R}^{k-1}} \delta_{k-1}(x_{t_1}, ..., x_{t_{k-1}}) \Big(\mathsf{E}_W\left[s(x_{t_k}, t_k)|x_{t_{k-1}}\right] - s(x_{t_{k-1}}, t_{k-1}) \Big) \prod_{j=1}^{k-1} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j})\, dx_{t_j}
\end{aligned}
$$

where we introduced the conditional expectation

$$
\mathsf{E}_W\left[f(\{x_s\}_{0 \le s \le T})|x_t\right] \;:=\; \int f(\{x_s\})\, dW(\{x_s\}_{t < s \le T}) \tag{8}
$$

with the obvious definition ($t = N_t \Delta t$)

$$
dW(\{x_t\}_{t < s \le T}) \;:=\; \lim_{\Delta t \to 0} \prod_{j=N_t+1}^{N_T} p_{\Delta t}(x_{(j-1)\Delta t}, x_{j\Delta t})\, dx_{j\Delta t} \tag{9}
$$

In particular, if the function f in (8) depends only on a single x_s, $f = f(x_s)$, then

$$
\mathsf{E}_W\left[f(x_s)|x_t\right] = \begin{cases} f(x_s) & \text{if } t \ge s \\ \displaystyle\int_{\mathbb{R}} f(x_s)\, p_{s-t}(x_t, x_s)\, dx_s & \text{if } t < s \end{cases} \tag{10}
$$

Let us compute the conditional expectation $\mathsf{E}_W\left[s(x_{t_k}, t_k)|x_{t_{k-1}}\right]$ in the last line of (7). We have, using $\Delta t = t_k - t_{k-1}$

$$
\begin{aligned}
\mathsf{E}_W\left[s(x_{t_k}, t_k)|x_{t_{k-1}}\right] &= \int_{\mathbb{R}} s(x_{t_k}, t_k)\, p_{t_k - t_{k-1}}(x_{t_{k-1}}, x_{t_k})\, dx_{t_k} \\
&= \tfrac{1}{\sqrt{2\pi\Delta t}} \int_{\mathbb{R}} S_0\, e^{\sigma x_{t_k} + (\mu - r - \sigma^2/2)t_k} e^{-\frac{(x_{t_k} - x_{t_{k-1}})^2}{2\Delta t}}\, dx_{t_k} \\
&= S_0\, e^{\sigma x_{t_{k-1}} + (\mu - r - \sigma^2/2)t_k}\, \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\sigma\sqrt{\Delta t}\, y} e^{-\frac{y^2}{2}}\, dy \\
&= S_0\, e^{\sigma x_{t_{k-1}} + (\mu - r - \sigma^2/2)t_k}\, e^{\frac{\sigma^2}{2}(t_k - t_{k-1})} \\
&= s(x_{t_{k-1}}, t_{k-1})\, e^{(\mu - r)(t_k - t_{k-1})}
\end{aligned} \tag{11}
$$

Now, suppose the factor $e^{(\mu-r)(t_k - t_{k-1})}$ in (11) would be absent. Then the round brackets in the third line of (7) would be zero for all k and the price of the option v_0 would be given by the expectation of the discounted payoff. Thus, we would be able to compute the price without knowing the hedging strategy, provided that there is a replicating strategy. Now, this factor is not absent but we can ask the following question: Is there some measure $d\tilde{W}$ such that

$$
\mathsf{E}_{\tilde{W}}\left[s(x_{t_k}, t_k)|x_{t_{k-1}}\right] \;=\; s(x_{t_{k-1}}, t_{k-1}) \tag{12}
$$

If this is the case then we can compute the price v_0 of the option with discounted payoff h by taking the expectation value with respect to $d\tilde{W}$,

$$v_0 \;=\; \mathsf{E}_{\tilde{W}}[\,h\,] \tag{13}$$

since the round brackets in the third line of (7) all vanish. There is the following

Theorem 9.1: Let $s_{t_k} = s(x_{t_k}, t_k)$ be a discounted geometric Brownian motion given by (2). Define the kernels $\tilde{p}_t(x,y) = \tilde{p}_t^{\mu,r,\sigma}(x,y)$ by

$$\tilde{p}_t(x,y) \;:=\; \tfrac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y-\frac{\mu-r}{\sigma}t)^2}{2t}} \tag{14}$$

Then:

a) The kernels $\tilde{p}$ satisfy

$$\int_{\mathbb{R}} \tilde{p}_s(x,y)\,\tilde{p}_t(y,z)\,dy \;=\; \tilde{p}_{s+t}(x,z) \tag{15}$$

and $\int_{\mathbb{R}} \tilde{p}_s(x,y)\,dy = 1$ for all x. That is, the measure

$$d\tilde{W}(\{x_t\}_{0<t\le T}) \;:=\; \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} \tilde{p}_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{16}$$

is well defined.

b) The price process (2) is a martingale with respect to $d\tilde{W}$. That is,

$$\begin{aligned}\mathsf{E}_{\tilde{W}}\big[s(x_{t'},t')\,|\,x_t\big] \;&:=\; \int s(x_{t'},t')\,d\tilde{W}(\{x_s\}_{t<s\le T}) \\ &=\; s(x_t,t) \quad \forall\, t<t'\end{aligned} \tag{17}$$

The measure $d\tilde{W}$ is called an equivalent martingale measure (with respect to the price process s_t).

Proof: a) Let $p_t(x,y)$ be the kernel of Chapter 4,

$$p_t(x,y) \;=\; \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}}$$

Then

$$\tilde{p}_t(x,y) = p_t\big(x - \tfrac{\mu-r}{\sigma}t, y\big) = p_t\big(x, y + \tfrac{\mu-r}{\sigma}t\big) = p_t\big(x - \tfrac{\mu-r}{2\sigma}, y + \tfrac{\mu-r}{2\sigma}\big) \tag{18}$$

such that, with Lemma 4.1,

$$\begin{aligned}\int_{\mathbb{R}} \tilde{p}_s(x,y)\,\tilde{p}_t(y,z)\,dy \;&=\; \int_{\mathbb{R}} p_s\big(x - \tfrac{\mu-r}{\sigma}s, y\big)\; p_t\big(y, z + \tfrac{\mu-r}{\sigma}t\big)\;dy \\ &=\; p_{s+t}\big(x - \tfrac{\mu-r}{\sigma}s, z + \tfrac{\mu-r}{\sigma}t\big) = \tilde{p}_{s+t}(x,z)\end{aligned} \tag{19}$$

which proves part (a). Part (b) is obtained as (11),

$$
\begin{aligned}
\mathsf{E}_{\tilde W}\big[s(x_{t'},t')\,|\,x_t\big] &= \int_{\mathbb{R}} s(x_{t'},t')\,\tilde p_{t'-t}(x_t,x_{t'})\,dx_{t'} \\
&= \frac{1}{\sqrt{2\pi(t'-t)}}\int_{\mathbb{R}} S_0\,e^{\sigma x_{t'}+(\mu-r-\sigma^2/2)t'}\,e^{-\frac{\left(x_t-x_{t'}-\frac{\mu-r}{\sigma}(t'-t)\right)^2}{2(t'-t)}}\,dx_{t'} \\
&= S_0\,e^{\sigma x_t+(\mu-r-\sigma^2/2)t}\,e^{-\frac{\sigma^2}{2}(t'-t)}\frac{1}{\sqrt{2\pi(t'-t)}}\int_{\mathbb{R}} e^{\sigma\left(x_{t'}-x_t+\frac{\mu-r}{\sigma}(t'-t)\right)}\,e^{-\frac{\left(x_t-x_{t'}-\frac{\mu-r}{\sigma}(t'-t)\right)^2}{2(t'-t)}}\,dx_{t'} \\
&= S_0\,e^{\sigma x_t+(\mu-r-\sigma^2/2)t}\,e^{-\frac{\sigma^2}{2}(t'-t)}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\sigma\sqrt{t'-t}\,y}\,e^{-\frac{y^2}{2}}\,dy \\
&= S_0\,e^{\sigma x_t+(\mu-r-\sigma^2/2)t} \\
&= s(x_t,t) \qquad (20)
\end{aligned}
$$

This proves the theorem. ■