Kapitel 5: Das Black-Scholes Modell als zeitstetiger Grenzwert des Binomialmodells, Teil2

Gestern haben wir gesehen, dass ein Binomialmodell mit Returns

$$\begin{aligned} \mathrm{ret}_{\mathrm{up}} &= \mu\Delta t + \sigma\sqrt{\Delta t} &(1)\\ \mathrm{ret}_{\mathrm{down}} &= \mu\Delta t - \sigma\sqrt{\Delta t} &(2)\end{aligned}$$

im Limes $\Delta t \to 0$ die Dynamik eines Black-Scholes Modells approximieren tut. Diese Tatsache benutzen wir jetzt, um Optionspreise im Black-Scholes Modell zu berechnen.

Now that we are in a position to approximate the Black-Scholes model with a suitable Binomial model, we can consider option prices and replicating strategies. Consider first the case of a european option with payoff $H = H(S_T)$ which depends only on the stock price at maturity. According to Theorem 3.2, its theoretical fair value V_0 is given by ($t_N = T$, $t_0 = 0$)

$$V_0 \;=\; e^{-rT}\sum_{k=0}^{N} H\big(S_0(1+\mathrm{ret}_{\mathrm{up}})^k(1+\mathrm{ret}_{\mathrm{down}})^{N-k}\big) \times \tbinom{N}{k} p_{\mathrm{rn}}^k (1-p_{\mathrm{rn}})^{N-k} \tag{3}$$

with the risk neutral probability

$$p_{\mathrm{rn}} \;=\; \frac{e^{r\Delta t} - 1 - \mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \tag{4}$$

and up- and down-returns given by

$$\begin{aligned} \mathrm{ret}_{\mathrm{up}} &= \mu\Delta t + \sigma\sqrt{\Delta t}\\ \mathrm{ret}_{\mathrm{down}} &= \mu\Delta t - \sigma\sqrt{\Delta t}\end{aligned}$$

Using $e^{r\Delta t} = 1 + r\Delta t + O\big((\Delta t)^2\big)$ and neglecting terms quadratic in Δt, we can write

$$p_{\mathrm{rn}} \;=\; \frac{(r-\mu)\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} \;=\; \frac{1 + \frac{r-\mu}{\sigma}\sqrt{\Delta t}}{2} \tag{5}$$

To obtain the option price under the Black-Scholes model, we have to calculate the $\Delta t \to 0$ limit of (3). A naive guess could be that the risk neutral probabilities converge actually to 1/2 and then (3) actually coincides with the real world expectation value (with $f = H$)

$$\mathsf{E}^{\mathrm{Bin}}\big[\,f(S_t)\,\big] \;=\; \sum_{k=0}^{N} f\Big(S_0(1+\mu\Delta t+\sigma\sqrt{\Delta t})^k(1+\mu\Delta t-\sigma\sqrt{\Delta t})^{N-k}\Big) \times \tfrac{1}{2^N}\tbinom{N}{k} \tag{6}$$

and this expression converges to the real world Black-Scholes expectation value

$$\begin{aligned} \mathsf{E}^{\mathrm{BS}}\big[\,f(S_t)\,\big] &= \int f\Big(S_0\,e^{(\mu-\frac{\sigma^2}{2})t\,+\,\sigma x_t}\Big)\,dW(\{x_t\}_{0<t\le T}) \\ &= \int_{\mathbb{R}} f\Big(S_0\,e^{(\mu-\frac{\sigma^2}{2})t\,+\,\sigma x_t}\Big)\,\tfrac{1}{\sqrt{2\pi t}}e^{-\frac{x_t^2}{2t}}\,dx_t \\ &= \int_{\mathbb{R}} f\Big(S_0\,e^{(\mu-\frac{\sigma^2}{2})t\,+\,\sqrt{t}\,\sigma x}\Big)\,\tfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \end{aligned} \tag{7}$$

If this would be true, it would be actually quite bad since in that case the option price would depend on the drift parameter μ and this parameter is basically not predictable. Knowing μ is basically equivalent to knowing whether the underlying is going up or down, this is not predictable. Recall that the basic result of the very elementary example in chapter 0 was that you have to buy half a stock and then you are save, regardless whether the underlying is going up or down.

Fortunately this is still true in the Black-Scholes model. The $\sqrt{\Delta t}$-term in the risk neutral probabilities is actually highly important and it has the effect that in the continuous time limit the drift parameter μ completely drops out of the pricing formula, it is simply substituted by the interest rate parameter r. There is the following

Theorem 5.2: Consider a Binomial model with returns (1,2) which converges to the Black-Scholes model with $dS_t/S_t = \mu dt + \sigma dx_t$ with real world drift parameter μ. Let V_0^{Bin} be the theoretical fair value of some european option $H = H(S_T)$ in the Binomial model. Then

$$\lim_{\Delta t\to 0} V_0^{\mathrm{Bin}} \;=\; V_0^{\mathrm{BS}} \tag{8}$$

where the theoretical fair value under the Black-Scholes model in given by

$$V_0^{\mathrm{BS}} \;=\; e^{-rT}\int_{\mathbb{R}} H\big(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}\big)e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \tag{9}$$

Proof: As in the proof of Theorem 5.1, we have

$$(1+\mathrm{ret_{up}})^k(1+\mathrm{ret_{down}})^{N-k} \;=\; e^{\sigma\sqrt{\Delta t}(2k-N)}\;e^{(\mu-\frac{\sigma^2}{2})T}\;e^{O(\sqrt{\Delta t})} \tag{10}$$

Abbreviating

$$\alpha \;:=\; \frac{r-\mu}{\sigma} \tag{11}$$

the risk neutral probability (5) is written as

$$p_{\mathrm{rn}} \;=\; \frac{1+\alpha\sqrt{\Delta t}}{2} \tag{12}$$

Using the Taylor expansion for $\log(1+x)$ again,

$$\begin{aligned} p_{\mathrm{rn}}^k(1-p_{\mathrm{rn}})^{N-k} &= \tfrac{1}{2^N}\,e^{k\log(1+\alpha\sqrt{\Delta t})+(N-k)\log(1-\alpha\sqrt{\Delta t})} \\ &= \tfrac{1}{2^N}\,e^{k(\alpha\sqrt{\Delta t}-\frac{\alpha^2}{2}\Delta t)+(N-k)(-\alpha\sqrt{\Delta t}-\frac{\alpha^2}{2}\Delta t)+O(\sqrt{\Delta t})} \\ &= \tfrac{1}{2^N}\,e^{-(N-2k)\alpha\sqrt{\Delta t}}e^{-\frac{\alpha^2}{2}T}e^{O(\sqrt{\Delta t})} \end{aligned} \tag{13}$$

Thus, the theoretical fair value (3) in the Binomial model becomes, again ignoring the last exponential $e^{O(\sqrt{\Delta t})}$ in (13),

$$\begin{aligned}
V_0 &= e^{-rT} \sum_{k=0}^{N} H\big(S_0 (1+\mathrm{ret}_{\mathrm{up}})^k (1+\mathrm{ret}_{\mathrm{down}})^{N-k}\big) \ \tbinom{N}{k} p_{\mathrm{rn}}^k (1-p_{\mathrm{rn}})^{N-k} \\
&= e^{-rT} \sum_{k=0}^{N} H\big(S_0 e^{\sigma\sqrt{\Delta t}(2k-N)} e^{(\mu-\frac{\sigma^2}{2})T}\big) \ \tbinom{N}{k} \tfrac{1}{2^N}\, e^{-(N-2k)\alpha\sqrt{\Delta t}} e^{-\frac{\alpha^2}{2}T} \\
&= e^{-rT} e^{-\frac{\alpha^2}{2}T} \sum_{k=0}^{N} \underbrace{H\big(S_0 e^{\sigma\sqrt{\Delta t}(2k-N)} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ e^{\alpha\sqrt{\Delta t}(2k-N)}}_{=:f[\sqrt{\Delta t}(2k-N)] \text{ for Lemma 5.1}} \ \tfrac{1}{2^N} \tbinom{N}{k} \\
&\overset{\Delta t \to 0}{\to} e^{-rT} e^{-\frac{\alpha^2}{2}T} \int_{\mathbb{R}} H\big(S_0 e^{\sigma x} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ e^{\alpha x}\ \tfrac{1}{\sqrt{2\pi T}}\ e^{-\frac{x^2}{2T}}\, dx \\
&= e^{-rT} \int_{\mathbb{R}} H\big(S_0 e^{\sigma x} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ \tfrac{1}{\sqrt{2\pi T}}\ e^{-\frac{1}{2}(\frac{x}{\sqrt{T}}-\sqrt{T}\alpha)^2}\, dx
\end{aligned}$$

Making the substitution of variables

$$y = \frac{x}{\sqrt{T}} - \sqrt{T}\alpha \quad \Leftrightarrow \quad x = \sqrt{T}y + T\alpha$$

we arrive at

$$\begin{aligned}
V_0 &= e^{-rT} \int_{\mathbb{R}} H\big(S_0 e^{\sigma x} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ \tfrac{1}{\sqrt{2\pi T}}\ e^{-\frac{1}{2}(\frac{x}{\sqrt{T}}-\sqrt{T}\alpha)^2} dx \\
& \quad e^{-rT} \int_{\mathbb{R}} H\big(S_0 e^{\sigma(\sqrt{T}y+T\alpha)} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ \tfrac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}y^2} dy \\
&= e^{-rT} \int_{\mathbb{R}} H\big(S_0 e^{\sigma\sqrt{T}y+T(r-\mu)} e^{(\mu-\frac{\sigma^2}{2})T}\big)\ \tfrac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}y^2} dy \\
&= e^{-rT} \int_{\mathbb{R}} H\big(S_0 e^{\sigma\sqrt{T}y} e^{(r-\frac{\sigma^2}{2})T}\big)\ \tfrac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}y^2} dy
\end{aligned}$$

which coincides with (9). ■

In the next chapter we apply the basic pricing formula (9),

$$V_0^{\mathrm{BS}} = e^{-rT} \int_{\mathbb{R}} H\Big(S_0\, e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x} \Big)\ e^{-\frac{x^2}{2}}\ \frac{dx}{\sqrt{2\pi}} \tag{9}$$

to standard call and put options and obtain in this way the famous Black-Scholes formulae, for which Robert Merton and Myron Scholes received the Economics Nobel Prize in 1997 (*"for a new method to determine the value of derivatives"*). Fischer Black died in 1995.