Hochschule RheinMain
Prof. Dr. D. Lehmann

Lösungen Übungsblatt 2 Ökonometrie

Aufgabe 1) a)

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx &= \left\{ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy \right\}^{1/2} \\
&= \left\{ \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}}\,d\varphi\, r\, dr \right\}^{1/2} \\
&= \left\{ 2\pi\, e^{-\frac{r^2}{2}} \Big|_0^{\infty} \right\}^{1/2} \\
&= \sqrt{2\pi}
\end{aligned}$$

b) Substituiere $y := \frac{x-\mu}{\sigma}$, $dy = dx/\sigma$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy \;=\; 1$$

Aufgabe 2) Es sei

$$p_{\mu,\sigma}(x) \;:=\; \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

a) Die Aussage " der Erwartungswert von ϕ ist μ " ist äquivalent zu

$$\mathsf{E}[\phi] \;=\; \int_{-\infty}^{\infty} \phi\, p_{\mu,\sigma}(\phi)\, d\phi \;=\; \mu\,.$$

Beweis:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi\, p_{\mu,\sigma}(\phi)\, d\phi &= \int_{-\infty}^{\infty} (\phi - \mu + \mu)\, p_{\mu,\sigma}(\phi)\, d\phi \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\phi-\mu) e^{-\frac{(\phi-\mu)^2}{2\sigma^2}}\, d\phi \;+\; \mu \times 1 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x\, e^{-\frac{x^2}{2\sigma^2}}\, dx \;+\; \mu \;=\; \mu\,,
\end{aligned}$$

das Integral ist 0 da der Integrand eine ungerade Funktion ist.

b) Die Aussage " die Varianz von ϕ ist σ^2 " ist äquivalent zu

$$\mathsf{V}[\phi] \;=\; \mathsf{E}\big[(\phi - \mathsf{E}[\phi])^2\big] \;=\; \int_{-\infty}^{\infty} (\phi-\mu)^2\, p_{\mu,\sigma}(\phi)\, d\phi \;=\; \sigma^2 \,.$$

Beweis:

$$\begin{aligned} \int_{-\infty}^{\infty} (\phi-\mu)^2\, p_{\mu,\sigma}(\phi)\, d\phi \;&=\; \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\phi-\mu)^2\, e^{-\frac{(\phi-\mu)^2}{2\sigma^2}}\, d\phi \\ &=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2\, y^2\, e^{-\frac{y^2}{2}}\, dy \;=\; \sigma^2 \end{aligned}$$

denn mit partieller Integration

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \times y\, e^{-\frac{y^2}{2}}\, dy \;&=\; \frac{1}{\sqrt{2\pi}} \Big\{ y \times (-e^{-\frac{y^2}{2}})\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-e^{-\frac{y^2}{2}})\, dy \Big\} \\ &=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy \;=\; 1 \,. \end{aligned} \tag{1}$$

c) Für $\mu = 0$ und $\sigma = 1$ haben wir

$$\begin{aligned} \mathsf{E}[\phi^2] \;&=\; \int_{-\infty}^{\infty} \phi^2\, e^{-\frac{\phi^2}{2}}\, \tfrac{d\phi}{\sqrt{2\pi}} \\ &\overset{\text{Gleichung (1) oben}}{=}\; 1 \end{aligned}$$

und

$$\begin{aligned} \mathsf{V}[\phi^2] \;&\overset{\text{Aufg.3a}}{=}\; \mathsf{E}[\phi^4] \;-\; \mathsf{E}[\phi^2]^2 \\ &=\; \int_{-\infty}^{\infty} \phi^4\, e^{-\frac{\phi^2}{2}}\, \tfrac{d\phi}{\sqrt{2\pi}} \;-\; 1^2 \\ &=\; \int_{-\infty}^{\infty} \phi^3 \times \phi\, e^{-\frac{\phi^2}{2}}\, \tfrac{d\phi}{\sqrt{2\pi}} \;-\; 1 \\ &=\; \int_{-\infty}^{\infty} \phi^3 \times \Big\{-\frac{d}{d\phi}\, e^{-\frac{\phi^2}{2}}\Big\}\, \tfrac{d\phi}{\sqrt{2\pi}} \;-\; 1 \\ &=\; \int_{-\infty}^{\infty} \Big\{\frac{d}{d\phi}\, \phi^3\Big\} \times\, e^{-\frac{\phi^2}{2}}\, \tfrac{d\phi}{\sqrt{2\pi}} \;-\; 1 \\ &=\; 3 \int_{-\infty}^{\infty} \phi^2\, e^{-\frac{\phi^2}{2}}\, \tfrac{d\phi}{\sqrt{2\pi}} \;-\; 1 \\ &=\; 3 \times 1 \;-\; 1 \;=\; 2 \,. \end{aligned}$$

Aufgabe 3) Wir haben

$$\begin{aligned}\mathsf{V}[X] &= \mathsf{E}\big[(X-\mathsf{E}[X])^2\big] \\ &= \mathsf{E}\big[\,X^2 - 2X\,\mathsf{E}[X] + \mathsf{E}[X]^2\,\big] \\ &= \mathsf{E}[\,X^2\,] - 2\,\mathsf{E}[X]\,\mathsf{E}[X] + \mathsf{E}[X]^2 \\ &= \mathsf{E}[X^2] - \mathsf{E}[X]^2\end{aligned}$$

und

$$\begin{aligned}\mathsf{Cov}[X,Y] &= \mathsf{E}\big[(X-\mathsf{E}[X])(Y-\mathsf{E}[Y])\big] \\ &= \mathsf{E}\Big[\,XY - \mathsf{E}[X]Y - X\mathsf{E}[Y] + \mathsf{E}[X]\mathsf{E}[Y]\,\Big] \\ &= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] - \mathsf{E}[X]\mathsf{E}[Y] + \mathsf{E}[X]\mathsf{E}[Y] \\ &= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]\end{aligned}$$