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Lösungen Übungsblatt 8
Finanzmathematik I

Aufgabe 1+2: We use theorem 4.1 from the lecture notes which was the following statement:

Theorem 4.1: *Let $F : \mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$. Then*

$$\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) = \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \tag{1}$$

where the p-functions on the right hand side of (1) are given by

$$p_\tau(x,y) \;:=\; \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \;.$$

Since all expectations and variances to be calculated depend on the Brownian motion observed at two times s and t with $s < t$, we have $m = 2$ for all integrals and

$$\mathsf{E}\big[F(x_s, x_t)\big] \;=\; \int_{\mathbb{R}^2} F(x_s, x_t)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \tag{2}$$

If the quantities to be calculated depend only on the difference $x_t - x_s$, that is, $F(x_s, x_t) = f(x_t - x_s)$ with some function f, we have furthermore

$$\begin{aligned}
\mathsf{E}\big[F(x_s, x_t)\big] &= \int_{\mathbb{R}^2} F(x_s, x_t)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(x_t - x_s)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(x_t - x_s)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(x_t - x_s)^2}{2(t-s)}}\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(y)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dx_s\, dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, dx_s \times \int_{\mathbb{R}} f(y)\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dy \\
&= 1 \times \int_{\mathbb{R}} f(\sqrt{t-s}\, v)\; e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;=\; \int_{\mathbb{R}} f(\sqrt{t-s}\, v)\; e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}}
\end{aligned} \tag{3}$$

Thus, with the formulae from Übungsblatt6, we obtain:

1a)

$$
\begin{aligned}
\mathsf{E}[\,x_t - x_s\,] &\overset{\substack{\text{Definiton}\\ \text{of } \mathsf{E}[\,\cdot\,]}}{=} \int (x_t - x_s)\, dW(\{x_t\}_{0<t\leq T}) \\
&\overset{(3)}{=} \int_{\mathbb{R}} \sqrt{t-s}\, v\, e^{-\frac{v^2}{2}}\, \frac{dv}{\sqrt{2\pi}} \;=\; 0
\end{aligned}
$$

1b)

$$
\begin{aligned}
\mathsf{E}[\,(x_t - x_s)^2\,] &\overset{\substack{\text{Definiton}\\ \text{of } \mathsf{E}[\,\cdot\,]}}{=} \int (x_t - x_s)^2\, dW(\{x_t\}_{0<t\leq T}) \\
&\overset{(3)}{=} \int_{\mathbb{R}} \big(\sqrt{t-s}\, v\big)^2\, e^{-\frac{v^2}{2}}\, \frac{dv}{\sqrt{2\pi}} \\
&= (t-s) \int_{\mathbb{R}} v^2\, e^{-\frac{v^2}{2}}\, \frac{dv}{\sqrt{2\pi}} \;=\; t-s
\end{aligned}
$$

which gives

$$
\begin{aligned}
\mathsf{V}[\,x_t - x_s\,] &= \mathsf{E}[\,(x_t - x_s)^2\,] \;-\; \big(\mathsf{E}[\,x_t - x_s\,]\big)^2 \\
&= t-s
\end{aligned}
$$

2a)

$$
\begin{aligned}
\mathsf{E}[\,|x_t - x_s|\,] &= \int |x_t - x_s|\, dW(\{x_t\}_{0<t\leq T}) \\
&\overset{(3)}{=} \int_{\mathbb{R}} \sqrt{t-s}\, |v|\, e^{-\frac{v^2}{2}}\, \frac{dv}{\sqrt{2\pi}} \\
&= 2\sqrt{t-s} \int_0^\infty v\, e^{-\frac{v^2}{2}}\, \frac{dv}{\sqrt{2\pi}} \\
&= 2\sqrt{t-s} \times \frac{1}{\sqrt{2\pi}} \;=\; \sqrt{\frac{2}{\pi}(t-s)}
\end{aligned}
$$

2b)

$$
\begin{aligned}
\mathsf{E}[\,|x_t - x_s|^2\,] &= \mathsf{E}[\,(x_t - x_s)^2\,] \\
&\overset{\text{Aufg.1b}}{=} t-s
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathsf{V}[\,|x_t - x_s|\,] &= \mathsf{E}[\,(x_t - x_s)^2\,] \;-\; \big(\mathsf{E}[\,|x_t - x_s|\,]\big)^2 \\
&= t - s - \tfrac{2}{\pi}(t-s) \;=\; \big(1 - \tfrac{2}{\pi}\big)(t-s)
\end{aligned}
$$

1c) Finally,

$$\begin{aligned}
\mathsf{Cov}[\ x_s\ ,\ x_t\ ] &= \mathsf{E}\big[(x_s - \mathsf{E}[x_s])(x_t - \mathsf{E}[x_t])\big] \\
&= \mathsf{E}\big[(x_s - 0)(x_t - 0)\big] \\
&= \mathsf{E}\big[x_s(x_t - x_s + x_s)\big] \\
&= \mathsf{E}\big[x_s(x_t - x_s)\big] \ + \ \mathsf{E}\big[x_s^2\big] \\
&= \qquad 0 \qquad\qquad + \quad s \quad = \quad s
\end{aligned}$$

since

$$\begin{aligned}
\mathsf{E}\big[x_s(x_t - x_s)\big] &\overset{(2)}{=} \int_{\mathbb{R}^2} x_s(x_t - x_s)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} x_s(x_t - x_s)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(x_t - x_s)^2}{2(t-s)}}\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} x_s\, y\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dx_s\, dy \\
&= \int_{\mathbb{R}} x_s\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, dx_s \times \int_{\mathbb{R}} y\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dy \\
&= 0 \times 0 \ = \ 0
\end{aligned}$$

and $\mathsf{E}\big[x_s^2\big] = s$ has been calculated in 1b.