Lösungen 5. Übungsblatt
Finanzmathematik I

Aufgabe 1: **a)** We have

$$S_N = S_0 \prod_{k=1}^{N} \bigl(1 + \mathrm{ret}_k\bigr)$$

and, since all the returns are independent,

$$\begin{aligned} \mathsf{E}[\,S_N\,] &= \mathsf{E}\Bigl[S_0 \prod_{k=1}^{N} \bigl(1 + \mathrm{ret}_k\bigr)\Bigr] \\ &= S_0 \prod_{k=1}^{N} \bigl(1 + \mathsf{E}[\mathrm{ret}_k]\bigr) \ . \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathsf{E}[\mathrm{ret}_k] &= +q \times p_{\mathrm{up}} \ + \ (-q) \times p_{\mathrm{down}} \\ &= q/2 - q/2 \ = \ 0 \end{aligned}$$

Thus,

$$\mathsf{E}[\,S_N\,] = S_0 \prod_{k=1}^{N} \bigl(1 + 0\bigr) \ = \ S_0 \ .$$

b) The notation $\mathsf{E}\bigl[\cdots \bigm| \{S_j\}_{j=0}^{k}\,\bigr]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned} S_N &= S_0 \prod_{j=1}^{N} \bigl(1 + \mathrm{ret}_j\bigr) \\ &= S_0 \prod_{j=1}^{k} \bigl(1 + \mathrm{ret}_j\bigr) \prod_{j=k+1}^{N} \bigl(1 + \mathrm{ret}_j\bigr) \\ &= S_k \prod_{j=k+1}^{N} \bigl(1 + \mathrm{ret}_j\bigr) \end{aligned}$$

and obtain

$$\begin{aligned} \mathsf{E}\bigl[\,S_N \bigm| \{S_j\}_{j=0}^{k}\,\bigr] &= \mathsf{E}\Bigl[\,S_k \prod_{j=k+1}^{N} \bigl(1 + \mathrm{ret}_j\bigr) \bigm| \{S_j\}_{j=0}^{k}\,\Bigr] \\ &= S_k \prod_{j=k+1}^{N} \bigl(1 + \mathsf{E}[\mathrm{ret}_j]\bigr) \\ &= S_k \prod_{j=k+1}^{N} \bigl(1 + 0\bigr) \ = \ S_k \ . \end{aligned}$$

c) Becaue of part (a), we have

$$\begin{aligned}
\mathsf{E}\big[\, \frac{1}{N}\sum_{m=1}^{N} S_m \,\big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big] \\
&= \frac{1}{N}\sum_{m=1}^{N} S_0 \;=\; S_0 \;.
\end{aligned}$$

d) Becaue of part (b), we obtain

$$\begin{aligned}
\mathsf{E}\Big[\, \frac{1}{N}\sum_{m=1}^{N} S_m \,\big|\, \{S_j\}_{j=0}^{k} \Big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^{k} \,\big] \\
&= \frac{1}{N}\Big\{ \sum_{m=1}^{k} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^{k} \,\big] \;+\; \sum_{m=k+1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^{k} \,\big] \Big\} \\
&= \frac{1}{N}\Big\{ \sum_{m=1}^{k} S_m \;+\; \sum_{m=k+1}^{N} S_k \Big\} \\
&= \frac{k}{N} \times \frac{1}{k}\sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N} \times S_k \;.
\end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N \;=\; S_0 \prod_{k=1}^{N} \big(1 + \mathrm{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} \;=\; \prod_{k=1}^{N} \frac{1}{1 + \mathrm{ret}_k} \;.$$

Since all the returns are independent,

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \,\big] &= \mathsf{E}\Big[\prod_{k=1}^{N} \frac{1}{1 + \mathrm{ret}_k} \Big] \\
&= \prod_{k=1}^{N} \mathsf{E}\Big[\frac{1}{1 + \mathrm{ret}_k} \Big] \\
&= \prod_{k=1}^{N} \Big\{ \frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2} \Big\} \\
&= \prod_{k=1}^{N} \Big\{ \frac{1}{1-q^2} \Big\} \\
&= \frac{1}{(1-q^2)^N} \;.
\end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots \big|\, \{S_j\}_{j=0}^{k}\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write as in part (b)

$$S_0 \,/\, S_N \;=\; S_0 \,/\, \Big\{ S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \,\big|\, \{S_j\}_{j=0}^k \big] &= S_0/S_k \; \mathsf{E}\Bigg[\prod_{m=k+1}^{N} \frac{1}{1+\mathrm{ret}_m} \,\Big|\, \{S_j\}_{j=0}^k \Bigg] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\Bigg[\frac{1}{1+\mathrm{ret}_m} \Bigg] \\
&= S_0/S_k \prod_{m=k+1}^{N} \Bigg\{ \frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2} \Bigg\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \Bigg\{ \frac{1}{1-q^2} \Bigg\} \\
&= S_0/S_k \; \frac{1}{(1-q^2)^{N-k}} \;.
\end{aligned}$$

Aufgabe 2: **a)** If the price process $\{S_k\}_{k=0}^N$ should be a martingale, we must have

$$\mathsf{E}\big[\, S_k \,\big|\, \{S_j\}_{j=0}^{k-1} \big] \;=\; S_{k-1}$$

for all k. Now we have

$$\begin{aligned}
\mathsf{E}\big[\, S_k \,\big|\, \{S_j\}_{j=0}^{k-1} \big] &= \mathsf{E}\big[\, S_{k-1}(1+\mathrm{ret}_k) \,\big|\, \{S_j\}_{j=0}^{k-1} \big] \\
&= S_{k-1}\big(1 + \mathsf{E}[\mathrm{ret}_k]\big) \\
&\overset{!}{=} S_{k-1}
\end{aligned}$$

which is the case if

$$\begin{aligned}
0 &\overset{!}{=} \mathsf{E}[\mathrm{ret}_k] \\
&= q \times p \;+\; \mathrm{ret}_{\mathrm{down}} \times (1-p)
\end{aligned}$$

which gives

$$\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{p}{1-p}\, q \;.$$

Thus, for $p = 3/4$ we get $\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{3/4}{1/4}\, q \;=\; -3q$.

b) Since the price process is a martingale by assumption, we can immediately say:

1a) $\mathsf{E}[\, S_N \,] \;=\; S_0$

1b) $\mathsf{E}\big[\, S_N \,\big|\, \{S_j\}_{j=0}^k \big] \;=\; S_k$

1c) $\mathsf{E}\big[\, \frac{1}{N} \sum_{m=1}^N S_m \,\big] \;=\; S_0$

1d) $\mathsf{E}\big[\,\frac{1}{N}\sum_{m=1}^{N} S_m \,\big|\, \{S_j\}_{j=0}^{k}\big] \;=\; \frac{k}{N} \times \frac{1}{k}\sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N} \times S_k\,.$

Furthermore, for the expectation of (1e) we obtain

$$\begin{aligned}
\mathsf{E}[\,S_0/S_N\,] &= \mathsf{E}\left[\prod_{k=1}^{N} \frac{1}{1+\mathrm{ret}_k}\right] \\
&= \prod_{k=1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_k}\right] \\
&= \prod_{k=1}^{N} \left\{\frac{1}{1+q} \times \frac{3}{4} + \frac{1}{1-3q} \times \frac{1}{4}\right\} \\
&= \prod_{k=1}^{N} \left\{\frac{3(1-3q)+1+q}{4(1+q)(1-3q)}\right\} \\
&= \left(\frac{1-2q}{1-2q-3q^2}\right)^{N}
\end{aligned}$$

and, in a similar way, for (1f)

$$\begin{aligned}
\mathsf{E}[\,S_0/S_N \,|\, \{S_j\}_{j=0}^{k}\,] &= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_m}\right] \\
&= S_0/S_k \left(\frac{1-2q}{1-2q-3q^2}\right)^{N-k}.
\end{aligned}$$