Lösungen 2. Übungsblatt
Finanzmathematik I

Aufgabe 2: **a)** Induction on p: Let $p = 1$. Since S_{ref} was equal to S_0, we have to have $S_{N_1} = S_0 \pm \Delta$ which gives

$$\delta_{N_1} \;=\; \frac{S_{N_1} - S_0}{\Delta} \;=\; \pm 1$$

which is obviously correct. Suppose the formula holds for p. Then, in order to have a trade event at N_{p+1}, we must have

$$S_{Np+1} - S_{N_p} \;=\; \pm\Delta$$

and the new δ is given by

$$\begin{aligned}
\delta_{N_{p+1}} &= \delta_{N_p} \;\pm\; 1 \\
&= \delta_{N_p} \;+\; \frac{S_{Np+1} - S_{N_p}}{\Delta} \\
&= \frac{S_{N_p} - S_0}{\Delta} \;+\; \frac{S_{Np+1} - S_{N_p}}{\Delta} \\
&= \frac{S_{Np+1} - S_0}{\Delta}\;.
\end{aligned}$$

b) Induction on p: For $p = 1$ the formula states

$$\begin{aligned}
V_{N_1} &= \frac{(S_{N_1} - S_0)^2 - \Delta^2}{2\Delta} \\
&= \frac{(\pm\Delta)^2 - \Delta^2}{2\Delta} \;=\; 0
\end{aligned}$$

which is correct since until time N_1 no position in the underlying has been taken. Suppose the formula is correct for p. Then

$$\begin{aligned}
V_{N_{p+1}} &= \sum_{j=1}^{N_{p+1}} \delta_{j-1}(S_j - S_{j-1}) \\
&= \sum_{j=1}^{N_p} \delta_{j-1}(S_j - S_{j-1}) \;+\; \sum_{j=N_p+1}^{N_{p+1}} \delta_{j-1}(S_j - S_{j-1}) \\
&= V_{N_p} \;+\; \sum_{j=N_p+1}^{N_{p+1}} \delta_{j-1}(S_j - S_{j-1})
\end{aligned}$$

For all j between N_p and N_{p+1} we have

$$\delta_j \;=\; \delta_{N_p} \;=\; \frac{S_{N_p} - S_0}{\Delta} \qquad \forall\; N_p \le j < N_{p+1}$$

Thus, using also the induction hypothesis,

$$\begin{aligned}
V_{N_{p+1}} &= V_{N_p} \;+\; \sum_{j=N_p+1}^{N_{p+1}} \delta_{N_p}(S_j - S_{j-1}) \\
&= V_{N_p} \;+\; \delta_{N_p} \sum_{j=N_p+1}^{N_{p+1}} (S_j - S_{j-1}) \\
&= V_{N_p} \;+\; \delta_{N_p}\,(S_{N_{p+1}} - S_{N_p}) \\
&= \frac{(S_{N_p} - S_0)^2 - p\Delta^2}{2\Delta} \;+\; \frac{S_{N_p} - S_0}{\Delta}\,(S_{N_{p+1}} - S_{N_p}) \\
&= \frac{(S_{N_p} - S_0)^2 - p\Delta^2 + 2(S_{N_p} - S_0)(S_{N_{p+1}} - S_{N_p})}{2\Delta} \\
&= \frac{(S_{N_p} - S_0)^2 - p\Delta^2 + 2(S_{N_p} - S_0)(S_{N_{p+1}} - S_{N_p}) + (S_{N_{p+1}} - S_{N_p})^2 - (S_{N_{p+1}} - S_{N_p})^2}{2\Delta} \\
&= \frac{(S_{N_p} - S_0)^2 - p\Delta^2 + 2(S_{N_p} - S_0)(S_{N_{p+1}} - S_{N_p}) + (S_{N_{p+1}} - S_{N_p})^2 - (\pm\Delta)^2}{2\Delta} \\
&= \frac{(S_{N_p} - S_0)^2 + 2(S_{N_p} - S_0)(S_{N_{p+1}} - S_{N_p}) + (S_{N_{p+1}} - S_{N_p})^2 - (p+1)\Delta^2}{2\Delta} \\
&= \frac{(S_{N_p} - S_0 + S_{N_{p+1}} - S_{N_p})^2 - (p+1)\Delta^2}{2\Delta} \\
&= \frac{(S_{N_{p+1}} - S_0)^2 - (p+1)\Delta^2}{2\Delta} \;.
\end{aligned}$$

c) If we put $\Delta = 1$, then there is a trade event on every day k, after N days we have N trade events, thus

$$V_N \;=\; \frac{(S_N - S_0)^2 - N\Delta^2}{2\Delta} \;=\; \frac{(S_N - S_0)^2 - N}{2} \;.$$