

**Lösungen Blatt 13**  
**Finanzmathematik I**

**1. Aufgabe:** Wird in der Übung vorgerechnet.

**2. Aufgabe:** We use the Ito-formula in integral form:

$$f(x_T, T) - f(x_0, 0) = \int_0^T \frac{\partial f}{\partial x}(x_t, t) dx_t + \int_0^T \left\{ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} \right\}(x_t, t) dt$$

or, since  $x_0 = 0$ ,

$$\int_0^T \frac{\partial f}{\partial x}(x_t, t) dx_t = f(x_T, T) - f(0, 0) - \int_0^T \left\{ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} \right\}(x_t, t) dt \quad (1)$$

a) We put  $f(x, t) = \frac{x^2}{2}$ . Then equation (1) becomes

$$\begin{aligned} \int_0^T x_t dx_t &= \frac{x_T^2}{2} - 0 - \int_0^T \left\{ \frac{1}{2} \times 1 + 0 \right\} dt \\ &= \frac{x_T^2}{2} - \frac{T}{2}. \end{aligned}$$

b) We put  $f(x, t) = \frac{x^3}{3}$ . Then equation (1) becomes

$$\begin{aligned} \int_0^T x_t^2 dx_t &= \frac{x_T^3}{3} - 0 - \int_0^T \left\{ \frac{1}{2} \times 2x_t + 0 \right\} dt \\ &= \frac{x_T^3}{3} - \int_0^T x_t dt. \end{aligned}$$

c) We put  $f(x, t) = tx$ . Then equation (1) becomes

$$\begin{aligned} \int_0^T t dx_t &= T x_T - 0 - \int_0^T \left\{ \frac{1}{2} \times 0 + x_t \right\} dt \\ &= T x_T - \int_0^T x_t dt. \end{aligned}$$

**3.Aufgabe:** Let

$$v(S, t) := e^{-rt} V(S, t)$$

Then, since  $v_0 = V_0$ ,

$$\begin{aligned} e^{-rT} H(S_T) - V_0 &= v(S_T, T) - v(S_0, 0) \\ &= \int_0^T dv \end{aligned}$$

where

$$\begin{aligned} dv &= d(e^{-rt} V) \\ &= d(e^{-rt}) V + e^{-rt} dV + d(e^{-rt}) dV \\ &= -r e^{-rt} dt V + e^{-rt} dV + 0 \end{aligned}$$

Now,

$$dV = \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS_t)^2 + \frac{\partial V}{\partial t} dt$$

Using

$$\delta_t = \frac{\partial V}{\partial S}(S_t, t)$$

and, recalling the calculation rules for Brownian motion  $(dx_t)^2 = dt$  and  $dx_t dt = (dt)^2 = 0$ ,

$$\begin{aligned} (dS_t)^2 &= (\mu S_t dt + \sigma S_t dx_t)^2 \\ &= 0 + 0 + \sigma^2 S_t^2 (dx_t)^2 \\ &= \sigma^2 S_t^2 dt \end{aligned}$$

we obtain

$$dV = \delta_t dS_t + \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt$$

We have to express  $dS_t$  through  $ds_t$  where  $s_t = e^{-rt} S_t$ . We have

$$\begin{aligned} ds_t &= d(e^{-rt} S_t) \\ &= -r e^{-rt} dt S_t + e^{-rt} dS_t \end{aligned}$$

or

$$e^{-rt} dS_t = ds_t + r e^{-rt} dt S_t$$

which gives

$$\begin{aligned} e^{-rt} dV &= \delta_t e^{-rt} dS_t + e^{-rt} \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt \\ &= \delta_t [ds_t + r e^{-rt} dt S_t] + e^{-rt} \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt \\ &= \delta_t ds_t + r S_t \frac{\partial V}{\partial S} dt + e^{-rt} \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt \end{aligned}$$

Thus,

$$\begin{aligned}
dv &= -r e^{-rt} dt V + e^{-rt} dV \\
&= -r e^{-rt} dt V + \delta_t ds_t + r S_t \frac{\partial V}{\partial S} dt + e^{-rt} \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt \\
&= \delta_t ds_t + e^{-rt} \left\{ \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - r V \right\} dt \\
&= \delta_t ds_t
\end{aligned}$$

where we used the Black-Scholes PDE in the last line. Thus,

$$\begin{aligned}
e^{-rT} H(S_T) - V_0 &= v(S_T, T) - v(S_0, 0) \\
&= \int_0^T dv \\
&= \int_0^T \delta_t ds_t .
\end{aligned}$$