

Lösungen Blatt 12 Finanzmathematik I

Aufgabe 1: In terms of the ϕ -variables, we have

$$\begin{aligned} I_1(\Delta t) &= \sum_{k=1}^N x_{t_{k-1}} (x_{t_k} - x_{t_{k-1}}) \\ &= \sum_{k=1}^N x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) \sqrt{\Delta t} \phi_k \end{aligned}$$

and

$$\begin{aligned} I_2(\Delta t) &= \sum_{k=1}^N \frac{x_{t_{k-1}} + x_{t_k}}{2} (x_{t_k} - x_{t_{k-1}}) \\ &= \frac{1}{2} I_1(\Delta t) + \frac{1}{2} \sum_{k=1}^N x_{t_k}(\phi_1, \dots, \phi_k) \sqrt{\Delta t} \phi_k \end{aligned}$$

Let us introduce the notation

$$\mathbb{E}_k[\cdot] = \int_{\mathbb{R}} \cdot e^{-\frac{\phi_k^2}{2}} \frac{d\phi_k}{\sqrt{2\pi}}$$

and

$$\mathbb{E}_{\{1, \dots, k\}}[\cdot] = \int_{\mathbb{R}^k} \cdot \prod_{j=1}^k e^{-\frac{\phi_j^2}{2}} \frac{d\phi_j}{\sqrt{2\pi}}$$

such that

$$\mathbb{E}_{\{1, \dots, k\}}[\cdot] = \mathbb{E}_{\{1, \dots, k-1\}}[\mathbb{E}_k[\cdot]]$$

Since

$$\begin{aligned} \mathbb{E}[I_1(\Delta t)] &= \mathbb{E}\left[\sum_{k=1}^N x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) \sqrt{\Delta t} \phi_k\right] \\ &= \sum_{k=1}^N \mathbb{E}\left[x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) \sqrt{\Delta t} \phi_k\right] \\ &= \sum_{k=1}^N \mathbb{E}_{\{1, \dots, k\}}\left[x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) \sqrt{\Delta t} \phi_k\right] \\ &= \sum_{k=1}^N \mathbb{E}_{\{1, \dots, k-1\}}\left[x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1})\right] \sqrt{\Delta t} \underbrace{\mathbb{E}_k[\phi_k]}_{=0} = 0 \end{aligned}$$

part (a) follows. To obtain part (b), observe that

$$x_{t_k} = x_{t_{k-1}} + \sqrt{\Delta t} \phi_k$$

and therefore

$$\begin{aligned} \mathbb{E}_{\{1, \dots, k\}} [x_{t_k}(\phi_1, \dots, \phi_k) \sqrt{\Delta t} \phi_k] &= \\ &= \mathbb{E}_{\{1, \dots, k\}} [\{x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) + \sqrt{\Delta t} \phi_k\} \sqrt{\Delta t} \phi_k] \\ &= \mathbb{E}_{\{1, \dots, k\}} [x_{t_{k-1}}(\phi_1, \dots, \phi_{k-1}) \sqrt{\Delta t} \phi_k] + \Delta t \mathbb{E}_{\{1, \dots, k\}} [\phi_k^2] \\ &= 0 + \Delta t \mathbb{E}_k [\phi_k^2] = \Delta t \times 1 \end{aligned}$$

which gives

$$\begin{aligned} \mathbb{E}[I_2(\Delta t)] &= \mathbb{E}\left[\frac{1}{2} I_1(\Delta t) + \frac{1}{2} \sum_{k=1}^N x_{t_k}(\phi_1, \dots, \phi_k) \sqrt{\Delta t} \phi_k\right] \\ &= \frac{1}{2} \sum_{k=1}^N \Delta t = \frac{N \Delta t}{2} = \frac{T}{2}. \end{aligned}$$

Aufgabe 2: The expectation of the Ito integral is always zero,

$$\begin{aligned} \mathbb{E}\left[\sum_{k=1}^N f(x_{t_{k-1}}) (x_{t_k} - x_{t_{k-1}})\right] &= \sum_{k=1}^N \mathbb{E}[f(x_{t_{k-1}}) (x_{t_k} - x_{t_{k-1}})] \\ &= \sum_{k=1}^N \mathbb{E}_{\{1, \dots, k-1\}} [f(x_{t_{k-1}})] \underbrace{\mathbb{E}_k [\sqrt{\Delta t} \phi_k]}_{=0} = 0. \end{aligned}$$

The expectation of the integral on the right hand side of (2) is

$$\begin{aligned} \frac{1}{2} \mathbb{E}\left[\int_0^T f'(x_t) dt\right] &= \frac{1}{2} \mathbb{E}\left[\int_0^T 3x_t^2 dt\right] \\ &= \frac{3}{2} \int_0^T \mathbb{E}[x_t^2] dt \\ &= \frac{3}{2} \int_0^T t dt \\ &= \frac{3}{2} \frac{t^2}{2} \Big|_0^T = \frac{3T^2}{4} \end{aligned}$$

To obtain the expectation value of the Stratonovich integral, we write again

$$x_{t_k} = x_{t_{k-1}} + \sqrt{\Delta t} \phi_k$$

such that

$$\frac{x_{t_{k-1}} + x_{t_k}}{2} = x_{t_{k-1}} + \frac{\sqrt{\Delta t} \phi_k}{2}$$

which gives

$$\left(\frac{x_{t_{k-1}}+x_{t_k}}{2}\right)^3 = x_{t_{k-1}}^3 + \frac{3}{2}x_{t_{k-1}}^2\sqrt{\Delta t}\phi_k + \frac{3}{4}x_{t_{k-1}}\Delta t\phi_k^2 + \frac{1}{8}\sqrt{\Delta t}^3\phi_k^3$$

Thus,

$$\begin{aligned} \mathsf{E}\left[f\left(\frac{x_{t_{k-1}}+x_{t_k}}{2}\right)(x_{t_k} - x_{t_{k-1}})\right] &= \mathsf{E}\left[\left(\frac{x_{t_{k-1}}+x_{t_k}}{2}\right)^3\sqrt{\Delta t}\phi_k\right] \\ &= \mathsf{E}\left[\left\{x_{t_{k-1}}^3 + \frac{3}{2}x_{t_{k-1}}^2\sqrt{\Delta t}\phi_k + \frac{3}{4}x_{t_{k-1}}\Delta t\phi_k^2 + \frac{1}{8}\sqrt{\Delta t}^3\phi_k^3\right\}\sqrt{\Delta t}\phi_k\right] \\ &= 0 + \frac{3}{2}\mathsf{E}_{\{1,\dots,k-1\}}[x_{t_{k-1}}^2]\Delta t\mathsf{E}_k[\phi_k^2] + 0 + \frac{1}{8}\Delta t^2\mathsf{E}_k[\phi_k^4] \\ &= \frac{3}{2}t_{k-1}\Delta t \times 1 + \frac{1}{8}\Delta t^2 \times 3 \end{aligned}$$

and we end up with

$$\begin{aligned} \mathsf{E}\left[\int_0^T f(x_t) \circ dx_t\right] &= \lim_{\Delta t \rightarrow 0} \sum_{k=1}^N \left\{ \frac{3}{2}t_{k-1}\Delta t + \frac{3}{8}\Delta t^2 \right\} \\ &= \frac{3}{2} \int_0^T t dt + 0 = \frac{3T^2}{4}. \end{aligned}$$