

Lösungen 10. Übungsblatt
Finanzmathematik I

Aufgabe 1: We have

$$\begin{aligned} V_0 &= e^{-rT} \int_{\mathbb{R}} H(S_T) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{\mathbb{R}} H(S_0 e^{\sigma\sqrt{T}x + (r-\sigma^2/2)T}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

Because of

$$\begin{aligned} S_T &\geq K \\ \Leftrightarrow S_0 e^{\sigma\sqrt{T}x + (r-\sigma^2/2)T} &\geq K \\ \Leftrightarrow \sigma\sqrt{T}x + (r - \sigma^2/2)T &\geq \log[K/S_0] \\ \Leftrightarrow \sigma\sqrt{T}x &\geq \log[K/S_0] - (r - \sigma^2/2)T \\ \Leftrightarrow x &\geq \frac{\log[K/S_0] - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ \Leftrightarrow x &\geq -\frac{\log[S_0/K] + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ \Leftrightarrow x &\geq -d_- \end{aligned}$$

we get

$$\begin{aligned} V_0 &= e^{-rT} \int_{\mathbb{R}} H(S_0 e^{\sigma\sqrt{T}x + (r-\sigma^2/2)T}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{-d_-}^{\infty} 1 e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{-\infty}^{d_-} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} N(d_-) . \end{aligned}$$

Aufgabe 2: Let us calculate more generally the price at some time $t \in [0, T]$. Then we have to use the following formula:

$$\begin{aligned}
V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma\sqrt{T-t}x + (r-\sigma^2/2)(T-t)}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}} \frac{S_0}{S_t e^{\sigma\sqrt{T-t}x + (r-\sigma^2/2)(T-t)}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \frac{S_0}{S_t} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}x - (r-\sigma^2/2)(T-t)} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \frac{S_0}{S_t} e^{-(r-\sigma^2/2)(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}x} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-2r(T-t)} \frac{S_0}{S_t} e^{\sigma^2/2(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}x} e^{-\frac{x^2}{2}} e^{-\frac{\sigma^2(T-t)}{2}} \frac{dx}{\sqrt{2\pi}} e^{\frac{\sigma^2(T-t)}{2}} \\
&= e^{(-2r+\sigma^2)(T-t)} \frac{S_0}{S_t} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^2} \frac{dx}{\sqrt{2\pi}} \\
&= e^{(-2r+\sigma^2)(T-t)} \frac{S_0}{S_t} \times 1 \\
&= \frac{S_0}{S_t} e^{(\sigma^2-2r)(T-t)}.
\end{aligned}$$