6. Abstract Manifolds

Up to this point we considered only embedded submanifolds of $\mathbb{R}^n$, i.e., manifolds which are *a priori* given as subsets of some Euclidean space $\mathbb{R}^n$. The concept of manifold can be generalized, however, to include abstractly given topological spaces with special properties (most importantly the property of being locally Euclidean). Even though almost all examples we will occupy ourselves with are, in fact, embedded submanifolds, the more general definition will give us more flexibility and enhance conceptual clarity.

(6.1) Definition. *A topological space M is called* **locally Euclidean** *of dimension d if for each point $p \in M$ there are an open neighborhood $U \subseteq M$ of p, an open subset $\Omega \subseteq \mathbb{R}^d$ and a homeomorphism $\psi : U \to \Omega$. In this case we call (U, ψ) a* **chart** *around p; the mapping ψ will be called a* **coordinate map**, *and its inverse $\psi^{-1} : \Omega \to U$ will be called a* **local parametrization** *of M around p. If $\psi(q) = \big(x_1(q), \ldots, x_d(q)\big)$ for $q \in U$, then the mappings $x_i : U \to \mathbb{R}$ are called* **local coordinates** *around p.*

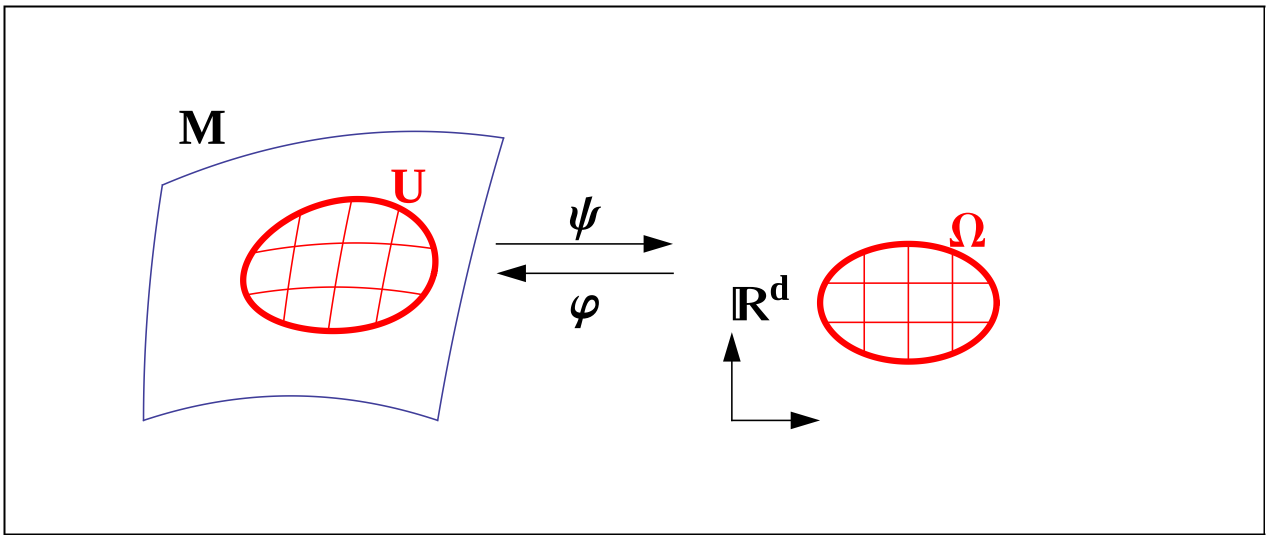

Fig. 6.1: Coordinate map ψ and local parametrization $\varphi = \psi^{-1}$ of a locally Euclidean space.

We would like to define the concept of “differentiability” or “smoothness” for functions $f : M \to \mathbb{R}$ defined on a locally Euclidean space. Since we know what differentiability means for functions $\mathbb{R}^n \to \mathbb{R}^m$, it is tempting to call a function $f : M \to \mathbb{R}$ smooth of class C^k, if for each chart (U, ψ) the mapping $f \circ \psi^{-1}$ is of class C^k. However, this is only a valid definition if it does not depend on the choice on the chart (U, ψ). This leads to the following definition.

(6.2) Definition. *Let M be a locally Euclidean space of dimension d. Two charts (U_1, ψ_1) and (U_2, ψ_2) are called C^k-***compatible** *if the* **transition map**

$$\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is a C^k-diffeomorphism. (This is meant to include the case $U_1 \cap U_2 = \emptyset$.)

Clearly, C^k-compatibility is an equivalence relation on the set of all charts of M.

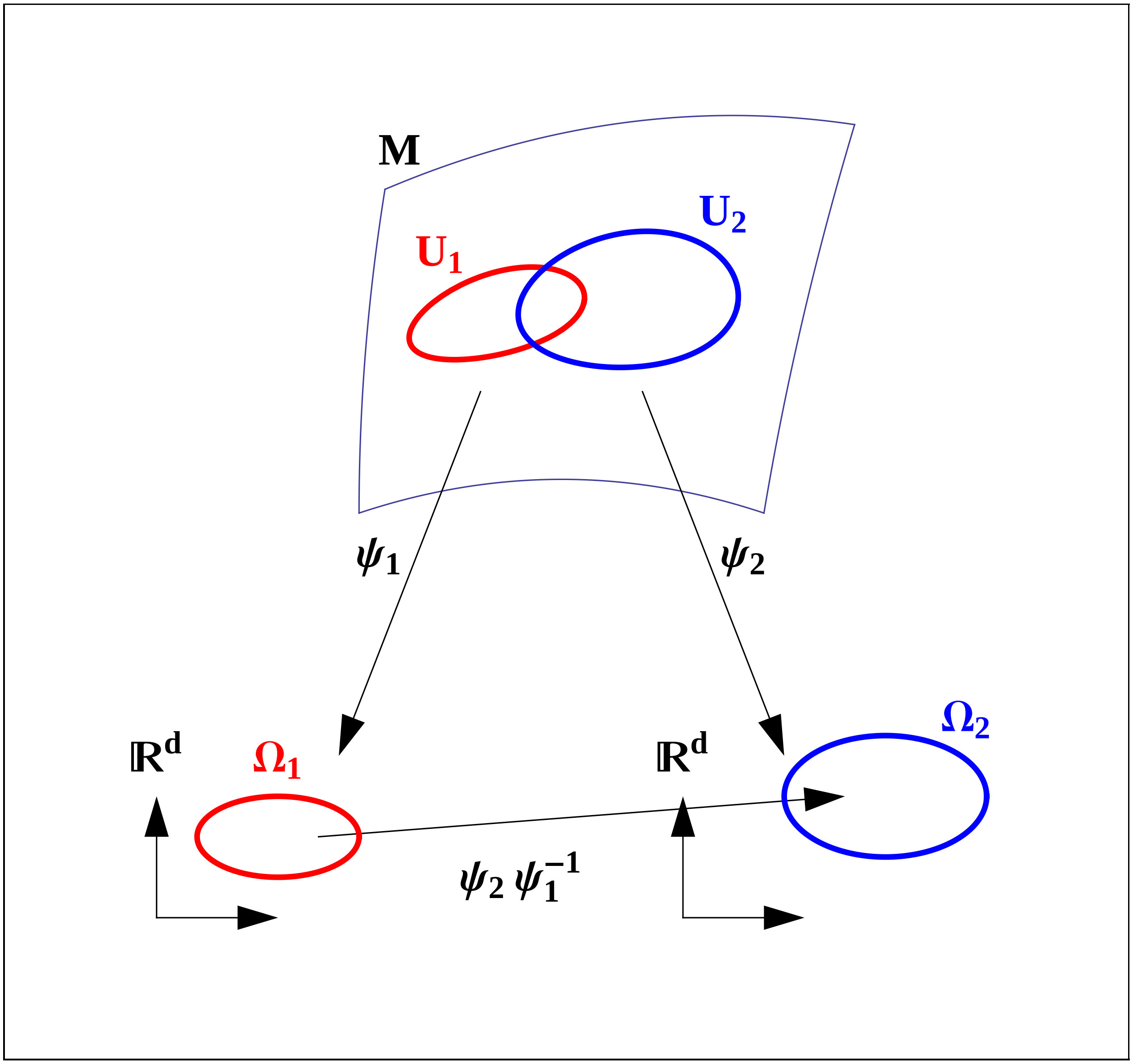

Fig. 6.2: Compatibility of coordinate charts.

Just like in everyday language an atlas is a collection of maps such that each point on the earth's surface is covered by at least one map in the atlas, we now define an atlas of a locally Euclidean space M as a collection of compatible charts such that each point of M is covered by at last one chart.

(6.3) Definition. *Let M be a locally Euclidean space of dimension d. A C^k-***atlas** *for M is a collection of charts $(U_i, \psi_i)_{i\in I}$ such that $\bigcup_{i\in I} U_i = M$ and such that any two charts (U_i, ψ_i) and (U_j, ψ_j) with $i, j \in I$ are C^k-compatible. Two such atlases are called C^k-compatible if each chart of one atlas is C^k-compatible with each chart of the other atlas. A* **differentiable structure** *of class C^k (or simply a C^k-structure) on M is an equivalence of atlases under C^k-compatibility.*

It is clear that the union of any family of C^k-compatible atlases is again an atlas compatible with each of the atlases in the family. Hence once a C^k-atlas $\mathfrak{A}$ has been specified for a locally Euclidean space M we can immediately enlarge $\mathfrak{A}$ by forming the union of $\mathfrak{A}$ with *all* other possible atlases which are C^k-compatible with $\mathfrak{A}$ (which is then clearly the **maximal atlas** C^k-compatible with $\mathfrak{A}$. Hence a differentiable structure on M can be identified with a maximal atlas for M.

(6.4) Definition. *A d-dimensional manifold of class C^k is a locally Euclidean Hausdorff space of dimension d which is equipped with a C^k-structure.*

Many authors require in addition the existence of a countable basis for the topology on M, which is helpful for certain constructions (and is automatically satisfied for embedded submanifolds of some space $\mathbb{R}^n$), but for the purposes of our text this condition will play no role. The Hausdorff property is included in the definition for convenience; it rules out certain pathological locally Euclidean spaces.

(6.5) Example. The sphere $\mathbb{S}^{n-1}$ can be covered with two charts using stereographic projection. Writing elements of $\mathbb{S}^{n-1}$ in the form $(\widehat{x}, x_n)$ where $\widehat{x} \in \mathbb{R}^{n-1}$ such that $\|\widehat{x}\|^2 + x_n^2 = 1$, we consider the northpole $N = (0, 1)$ and the south pole $S = (0, -1)$. Then $\mathbb{S}^{n-1}$ is covered by the charts (U_1, ψ_1) and (U_2, ψ_2) where

$$U_1 := \mathbb{S}^{n-1} \setminus \{N\}, \qquad \psi_1(\widehat{x}, x_n) := \frac{1}{1-x_n}\widehat{x}, \qquad \psi_1^{-1}(\xi) = \frac{1}{1+\|\xi\|^2}\begin{bmatrix} 2\xi \\ \|\xi\|^2 - 1 \end{bmatrix}$$

and

$$U_2 := \mathbb{S}^{n-1} \setminus \{S\}, \qquad \psi_2(\widehat{x}, x_n) := \frac{1}{1+x_n}\widehat{x}, \qquad \psi_2^{-1}(\xi) = \frac{1}{1+\|\xi\|^2}\begin{bmatrix} 2\xi \\ 1 - \|\xi\|^2 \end{bmatrix}.$$

We have $\psi_1(U_1 \cap U_2) = \psi_2(U_1 \cap U_2) = \mathbb{R}^{n-1} \setminus \{0\}$, and $\psi_2 \circ \psi_1^{-1} : \mathbb{R}^{n-1} \setminus \{0\} \to \mathbb{R}^{n-1} \setminus \{0\}$ is given by $(\psi_2 \circ \psi_1^{-1})(\xi) = \xi / \|\xi\|^2$ and hence is of class C^∞ (even C^ω). Thus the atlas consisting of the two charts (U_1, ψ_1) and (U_2, ψ_2) equips $\mathbb{S}^{n-1}$ with a C^∞- and even a C^ω-structure.

(6.6) Example. Let $M \subseteq \mathbb{R}^n$ be a d-dimensional embedded submanifold of $\mathbb{R}^n$ of class C^k. By the very definition of an embedded submanifold, each point $p \in M$ possesses an open neighborhood U which can be parametrized by a C^k-mapping $\varphi : \Omega \to \mathbb{R}^n$ such that Ω is an open subset of $\mathbb{R}^d$ and $\varphi(\Omega) = U$. The open sets U on M defined in this way clearly provide a C^k-atlas for M. Hence every embedded submanifold as defined in (1.6) is also a manifold in the abstract sense of (6.4).

(6.7) Example. We define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by declaring v and w equivalent if and only if there is a positive number $\lambda > 0$ such that $w = \lambda v$; the equivalence class of a vector $v \in \mathbb{R}^{n+1} \setminus \{0\}$ is denoted by $[v]$. If $v = (v_0, v_1, \ldots, v_n)$ is given in coordinates, we usually write $[v_0 : v_1 : \cdots : v_n]$ instead of $[v]$. The quotient space $\mathbb{R}^{n+1}/\sim$, equipped with the quotient topology, is denoted by $\mathbb{P}^n(\mathbb{R}) := \big\{[v] \mid v \in \mathbb{R}^{n+1}\setminus\{0\}\big\}$. (This is called the n-dimensional real projective space. Note that $\mathbb{P}^n(\mathbb{R})$ is not given as a subset of some Euclidean space, but rather as an abstract topological space.) We claim that $\mathbb{P}^n(\mathbb{R})$ is covered by the charts (U_0,ψ_0), (U_1,ψ_1), …, (U_n,ψ_n) where $U_i := \{[x_0:\cdots:x_n] \mid x_i \neq 0\}$ and where $\psi_i : U_i \to \mathbb{R}^n$ is given by

$$\psi_i([x_0:x_1:\cdots:x_n]) \;:=\; \left(\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right).$$

Note that U_i and ψ_i are well-defined and that the inverse map $\varphi_i := \psi_i^{-1} : \mathbb{R}^n \to U_i$ is given by

$$\varphi_i(u_1,\ldots,u_n) \;=\; [u_1:\ldots:u_{i-1}:1:u_i:\cdots:u_n].$$

The transition maps $\psi_j \circ \psi_i^{-1} = \varphi_j^{-1}\circ\varphi_i$ are given for $j>i$ by

$$(\varphi_j^{-1}\circ\varphi_i)(u_1,\ldots,u_n) \;=\; \left(\frac{u_1}{u_j},\ldots,\frac{u_{i-1}}{u_j},\frac{1}{u_j},\frac{u_i}{u_j},\ldots,1,\ldots,\frac{u_n}{u_j}\right)$$

and by an analogous formula for $j<i$; they are clearly of class C^∞ and even of class C^ω. Hence $\mathbb{P}^n(\mathbb{R})$ is an n-dimensional manifold of class C^ω. For clarity's sake, we write down all transition maps in the special case $n=3$ in which we have $\varphi_0(u,v)=[1:u:v]$, $\varphi_1(u,v)=[u:1:v]$, $\varphi_2(u,v)=[u:v:1]$ and

$$\psi_0([x:y:z]) = \left(\frac{y}{x},\frac{z}{x}\right),\quad \psi_1([x:y:z]) = \left(\frac{x}{y},\frac{z}{y}\right),\quad \psi_2([x:y:z]) = \left(\frac{x}{z},\frac{y}{z}\right)$$

and hence

$$\begin{aligned}
(\varphi_1^{-1}\circ\varphi_0)(u,v) &= \left(\frac{1}{u},\frac{v}{u}\right), &\qquad (\varphi_2^{-1}\circ\varphi_0)(u,v) &= \left(\frac{1}{v},\frac{u}{v}\right),\\
(\varphi_2^{-1}\circ\varphi_1)(u,v) &= \left(\frac{u}{v},\frac{1}{v}\right), &\qquad (\varphi_0^{-1}\circ\varphi_1)(u,v) &= \left(\frac{1}{u},\frac{v}{u}\right),\\
(\varphi_0^{-1}\circ\varphi_2)(u,v) &= \left(\frac{v}{u},\frac{1}{u}\right), &\qquad (\varphi_1^{-1}\circ\varphi_2)(u,v) &= \left(\frac{u}{v},\frac{1}{v}\right).
\end{aligned}$$

(6.8) Example. Every open subset Ω of a d-dimensional manifold M of class C^k inherits from M a manifold structure. Namely, if $\mathfrak{A}$ is a C^k-atlas for M then $\mathfrak{A}_0 := \{(U\cap\Omega,\psi|_{U\cap\Omega}) \mid (U,\psi)\in\mathfrak{A}\}$ is a C^k-atlas for Ω. With the differential structure given by this atlas (and with the subspace topology inherited from M) we call Ω an **open submanifold** of M.

(6.9) Example. Let M_1 and M_2 be C^k-manifolds of dimensions d_1 and d_2, respectively. Given a C^k-atlas $\mathfrak{A}_1$ for M_1 and a C^k-atlas $\mathfrak{A}_2$ for M_2, a C^k-atlas for M is given by $\mathfrak{A} := \{(U_1\times U_2,\psi_1\times\psi_2) \mid (U_1,\psi_1)\in\mathfrak{A}_1,\ (U_2,\psi_2)\in\mathfrak{A}_2\}$. Equipped with the differentiable structure defined by this atlas and with the product topology, we call M the **product manifold** of M_1 and M_2. This is a C^k-manifold of dimension d_1+d_2.

We are now in a position to define what it means for a mapping between locally Euclidean spaces to be smooth.

(6.10) Definition. *Let M and N be locally Euclidean spaces of dimensions m and n, respectively, both equipped with a C^k-structure. A mapping $f: M \to N$ is said to be of class C^k if it is continuous and if for all charts (U,α) in M and (V,β) in N such that $f(U) \subseteq V$ the mapping $\beta \circ f \circ \alpha^{-1} : \alpha(U) \to \beta(V)$ is of class C^k. (We call $\beta \circ f \circ \alpha^{-1}$ the* **coordinate representation** *of f with respect to the chosen charts U and V).*

The continuity of f ensures that, given a point $p \in M$ and a coordinate chart V around $f(p) \in N$, there is a coordinate chart U around p satisfying $f(U) \subseteq V$.

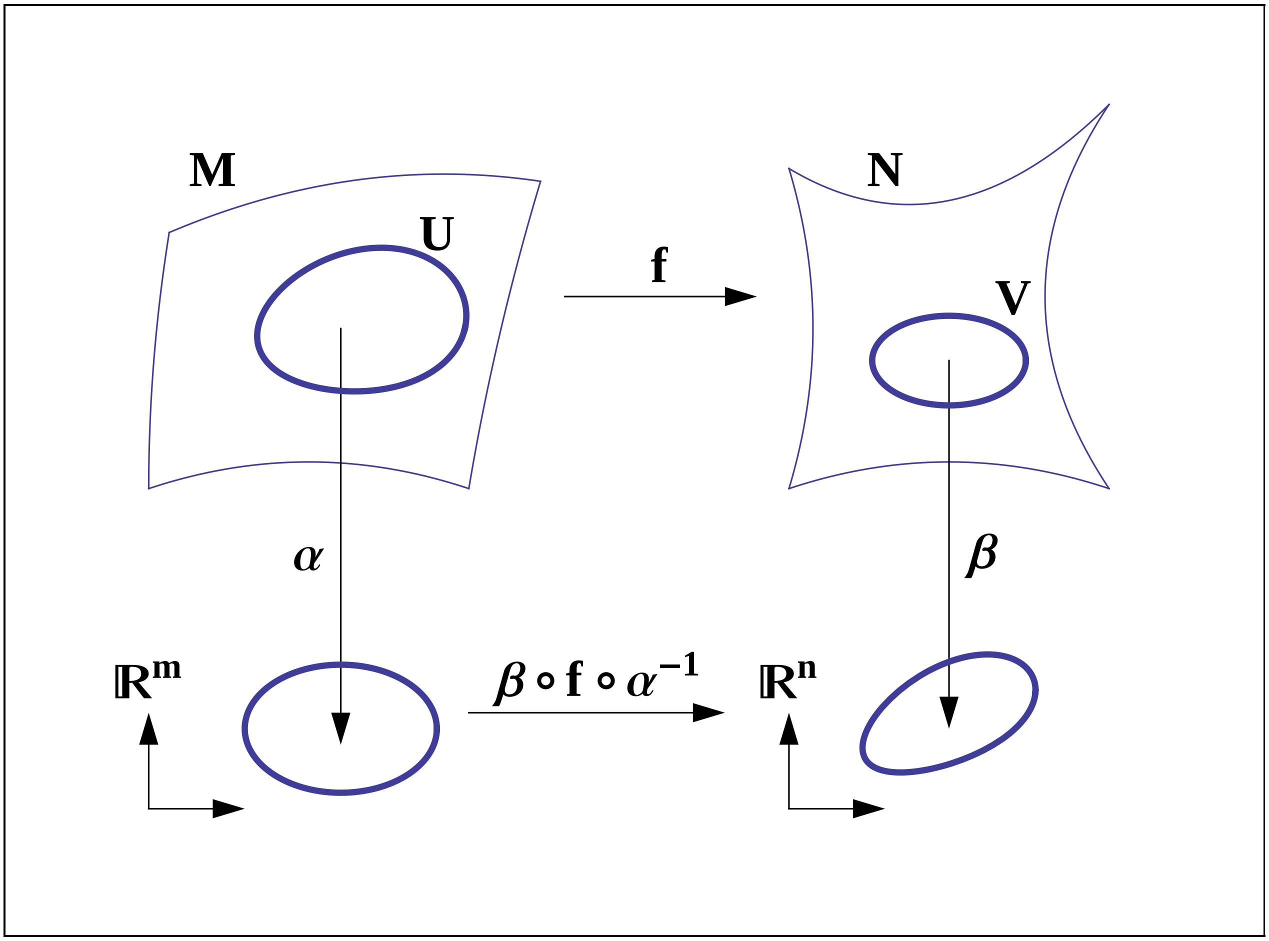

Fig. 6.3: Smoothness of a mapping between locally Euclidean spaces.

We now proceed to define the tangent space at a point of an abstractly defined manifold. For an embedded submanifold of M of $\mathbb{R}^n$ we defined a tangent vector at a point $p \in M$ as the velocity vector $\alpha'(0)$ of a curve in M satisfying $\alpha(0) = p$. Since taking the derivative $\alpha'(0)$ takes place in the ambient space $\mathbb{R}^n$, this definition does not make sense for an abstract manifold M for which something like an "ambient space" does not exist. However, for an embedded submanifold we can clearly say when two curves α_1 and α_2 through p are equivalent in the sense that they yield the same tangent vector at p, and the condition of equivalence can be expressed in a way which makes sense for abstract manifolds. This leads to the idea of simply identifying a tangent vector with an equivalence class of curves.

(6.11) Definition. *Let M be a C^k-manifold of dimension d and let $p \in M$ be a point in M. Two curves $\alpha_1, \alpha_2 : (-\varepsilon, \varepsilon) \to M$ with $\alpha_1(0) = \alpha_2(0) = p$ are called equivalent if $(\psi \circ \alpha_1)'(0) = (\psi \circ \alpha_2)'(0)$ for each chart (U, ψ) around p. The* **tangent space** *T_pM of M at p is the set of the equivalence classes of curves through p under this equivalence relation. The* **tangent bundle** *TM of M is the set of all pairs (p, v) where $p \in M$ and $v \in T_pM$.*

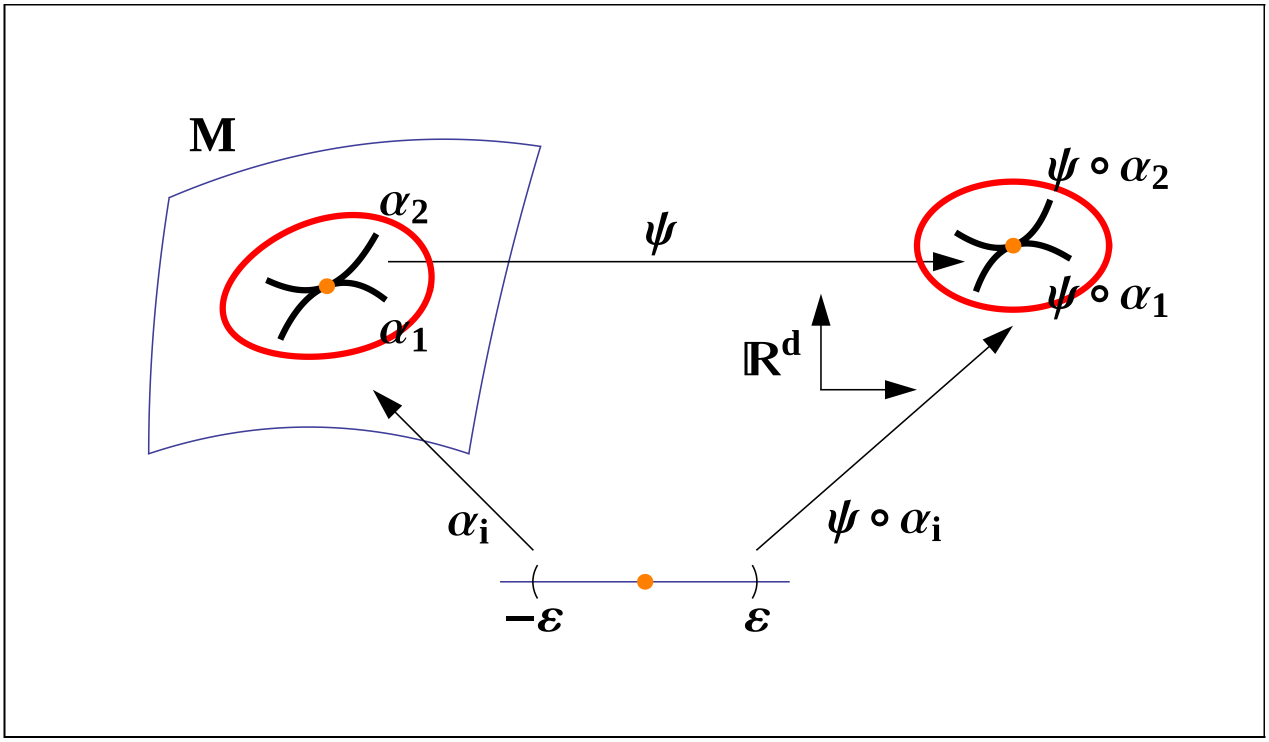

Fig. 6.4: Definition of tangent vectors as equivalence classes of curves.

If $[\alpha]$ denotes the equivalence class of a curve α through p and if ψ is any coordinate map around p, the mapping

$$\Phi : \begin{array}{ccc} T_pM & \to & \mathbb{R}^d \\ [\alpha] & \mapsto & (\psi \circ \alpha)'(0) \end{array}$$

is well-defined and bijective. Then T_pM becomes a vector space by declaring this mapping to be an isomorphism of vector spaces, i.e., by defining addition and scalar multiplication on T_pM by

$$[\alpha] \oplus [\beta] := \Phi^{-1}\big(\Phi([\alpha]) + \Phi([\beta])\big) \qquad \text{and} \qquad \lambda \odot [\alpha] := \Phi^{-1}\big(\lambda\Phi([\alpha])\big).$$

(It is readily checked that this definition is independent of the particular choice of the coordinate map ψ.) Hence T_pM is a d-dimensional vector space. A (coordinate-dependent) basis of T_pM is given by the tangent vectors

$$\left.\frac{\partial}{\partial x_i}\right|_p \;:=\; \Phi^{-1}(e_i) \;=\; \text{equivalence class of the curve } \alpha(t) := \psi^{-1}\big(\psi(p)+te_i\big)$$

where $1 \leq i \leq d$.

(6.12) Remark. If $\Omega \subseteq \mathbb{R}^m$ is an open subset of $\mathbb{R}^m$ and if $p \in \Omega$, then the mapping $T_p\Omega \to \mathbb{R}^m$ given by $[\alpha] \mapsto \alpha'(0)$ is well-defined and an isomorphism; hence we can (and shall) always identify $T_p\Omega$ with $\mathbb{R}^m$.

Let M and N be C^k-manifolds and let $f : M \to N$ be a mapping of class C^k. Given a point $p \in M$, let us consider the above equivalence relation $\sim$ of curves in M through p and the corresponding equivalence relation $\approx$ of curves in N through $f(p)$. It is readily checked that $\alpha_1 \sim \alpha_2$ implies $f \circ \alpha_1 \approx f \circ \alpha_2$; hence the following definition is possible.

(6.13) Definition. *Let $f : M \to N$ be a C^k-mapping between two C^k-manifolds and let $p \in M$. Then the mapping*

$$f'(p) : \begin{array}{ccc} T_pM & \to & T_{f(p)}N \\ [\alpha] & \mapsto & [f \circ \alpha] \end{array}$$

is well-defined and is called the **derivative** *of f at p.*

It is readily checked that $f'(p)$ is a linear mapping. The special case $f : M \to \mathbb{R}$ is of particular interest. In this case $f'(p) : T_pM \to \mathbb{R}$ is a linear form on T_pM, i.e., an element of the dual space $(T_pM)^\star$.

(6.14) Remark. *Let (U, ψ) be a coordinate chart of a manifold M, say $\psi(p) = \big(x_1(p), \cdots, x_d(p)\big) \in \mathbb{R}^d$ for $p \in U$. Then $\big(x_1'(p), \ldots, x_d'(p)\big)$ is the basis of $(T_pM)^\star$ which is dual to $\big(\partial/\partial x_1|_p, \ldots, \partial/\partial x_d|_p\big)$. Moreover, if $f : U \to \mathbb{R}$ is of class C^1 and if $f = F \circ \psi = F(x_1, \ldots, x_d)$ then*

$$f'(p) \;=\; \sum_{i=1}^d \frac{\partial F}{\partial x_i}\big(\psi(p)\big)x_i'(p).$$

Proof. We have $x_i = p_i \circ \psi$ where $p_i : \mathbb{R}^d \to \mathbb{R}$ is the projection onto the i-th component. Hence

$$x_i'(p)\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) \;=\; x_i'(p)\left[\psi^{-1}\big(\psi(p)+te_j\big)\right] \;=\; \left[x_i \circ \psi^{-1}\big(\psi(p)+te_j\big)\right]$$

$$=\; \left[p_i\big(\psi(p)+te_j\big)\right] \;=\; \left[x_i(p)+t\,p_i(e_j)\right] \;=\; \delta_{ij} \;=\; \begin{cases} 1, & \text{if } i=j; \\ 0, & \text{if } i \neq j. \end{cases}$$

If $f = F \circ \psi$ and if $\alpha : (-\varepsilon, \varepsilon) \to M$ is any C^1-curve with $\alpha(0) = p$ then

$$f'(p)[\alpha] = [f \circ \alpha] = \big[F\big(x_1(\alpha(t)), \ldots, x_d(\alpha(t))\big)\big] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F\big(x_1(\alpha(t)), \ldots, x_d(\alpha(t))\big)$$
$$= \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}\big(\psi(p)\big)[x_i \circ \alpha] = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}\big(\psi(p)\big) x_i'(p)[\alpha].$$

Since α was arbitrarily chosen this means $f'(p) = \sum_{i=1}^{d} (\partial F/\partial x_i)\big(\psi(p)\big)x_i'(p)$. ■

As in Chapter 2, we can form the tangent bundle $TM = \bigcup_{p \in M} T_pM$. We want to equip TM with both a topology and a differentiable structure.

(6.15) Theorem and Definition. *Let M be a d-dimensional manifold of class C^k. With each local parametrization $\varphi : \Omega \to M$ we associate the set $\widehat{\Omega} := \Omega \times \mathbb{R}^d$ and the mapping $\widehat{\varphi} : \widehat{\Omega} \to TM$ given by*

$$\widehat{\varphi}(u, w) := \big(\varphi(u), \varphi'(u)w\big)$$

whose inverse is given by $\widehat{\varphi}^{-1}(p, v) = \big(\varphi^{-1}(p), \varphi'\big(\varphi^{-1}(p)\big)^{-1} v\big)$. If $\mathfrak{A}$ is an atlas of class C^k for M and if $\mathfrak{B}$ is the family of local parametrizations associated with this atlas, then $\{\Omega \times \mathbb{R}^d \mid (\Omega, \varphi) \in \mathfrak{B}\}$ is the basis of a topology on TM, and

$$\widehat{\mathfrak{B}} := \{(\widehat{\Omega}, \widehat{\varphi}) \mid (\Omega, \varphi) \in \mathfrak{B}\}$$

gives rise to an atlas of class C^{k-1} for TM. Consequently, TM carries the structure of a $2d$-dimensional manifold of class C^{k-1}. With this manifold structure we call TM the **tangent bundle** *of M.*

Proof. We only need to check the smoothness of the transition maps. Given $u \in \Omega$ and $w \in \mathbb{R}^d$ we have

$$\begin{aligned}
(\widehat{\varphi}_2^{-1} \circ \widehat{\varphi}_1)(u, w) &= \widehat{\varphi}_2^{-1}\big(\varphi_1(u), \varphi_1'(u)w\big) \\
&= \Big(\varphi_2^{-1}\big(\varphi_1(u)\big),\, \varphi_2'\big(\varphi_2^{-1}(\varphi_1(u))\big)^{-1} \varphi_1'(u)w\Big) \\
&= \Big(\big(\varphi_2^{-1} \circ \varphi_1\big)(u),\, \varphi_2'\big((\varphi_2^{-1} \circ \varphi_1)(u)\big)^{-1} \varphi_1'(u)w\Big) \\
&= \Big(\big(\varphi_2^{-1} \circ \varphi_1\big)(u),\, (\varphi_2^{-1})'\big(\varphi_1(u)\big)\varphi_1'(u)w\Big) \\
&= \Big(\big(\varphi_2^{-1} \circ \varphi_1\big)(u),\, (\varphi_2^{-1} \circ \varphi_1)'(u)w\Big),
\end{aligned}$$

where we used the fact that $(\varphi_2^{-1})'(x) = \varphi_2'\big(\varphi_2^{-1}(x)\big)^{-1}$ in the step from the third to the fourth line. (This fact follows by taking derivatives on both sides of the identity $(\varphi_2 \circ \varphi_2^{-1})(x) = x$ using the chain rule, which yields $\varphi_2'\big(\varphi_2^{-1}(x)\big)\,(\varphi_2^{-1})'(x) = \mathbf{1}$.) This shows that if $\varphi_2^{-1} \circ \varphi_1$ is of class C^k then $\widehat{\varphi}_2^{-1} \circ \widehat{\varphi}_1$ is of class C^{k-1}. ■

It is now readily seen that a C^k-mapping $f : M \to N$ between C^k-manifolds induces a C^{k-1}-mapping $f_\star : TM \to TN$ via $f_\star(p,v) := \big(f(p), f'(p)v\big)$. As a straightforward consequence of the chain rule we see that if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ then $(g \circ f)_\star = g_\star \circ f_\star$. In a way completely analogous to (6.15), we now equip the cotangent bundle $T^\star M = \bigcup_{p\in M} (T_pM)^\star$ with a manifold structure.

(6.16) Theorem and Definition. *Let M be a d-dimensional manifold of class C^k. With each local parametrization $\varphi : \Omega \to M$ we associate the set $\widehat{\Omega} := \Omega \times (\mathbb{R}^d)^\star$ and the mapping $\widehat{\varphi} : \widehat{\Omega} \to T^\star$ given by*

$$\widehat{\varphi}(u,a) \;:=\; \big(\varphi(u), a \circ \varphi'(u)^{-1}\big)$$

whose inverse is given by $\widehat{\varphi}^{-1}(p,\lambda) = \big(\varphi^{-1}(p), \lambda \circ \varphi'\big(\varphi^{-1}(p)\big)\big)$. If $\mathfrak{A}$ is an atlas of class C^k for M and if $\mathfrak{B}$ is the family of local parametrizations associated with this atlas, then $\{\Omega \times (\mathbb{R}^d)^\star \mid (\Omega,\varphi) \in \mathfrak{B}\}$ is the basis of a topology on TM, and

$$\widehat{\mathfrak{B}} \;:=\; \{(\widehat{\Omega}, \widehat{\varphi}) \mid (\Omega,\varphi) \in \mathfrak{B}\}$$

gives rise to an atlas of class C^{k-1} for $T^\star M$. Consequently, $T^\star M$ carries the structure of a $2d$-dimensional manifold of class C^{k-1}. With this manifold structure we call $T^\star M$ the **cotangent bundle** *of M.*

Proof. Again, we only need to check the smoothness of the transition maps. Given $u \in \Omega$ and $a \in (\mathbb{R}^d)^\star$ we have

$$\begin{aligned}
&(\widehat{\varphi}_2^{-1} \circ \widehat{\varphi}_1)(u,a) \;=\; \widehat{\varphi}_2^{-1}\big(\varphi_1(u),\, a \circ \varphi_1'(u)^{-1}\big)\\
&= \big(\varphi_2^{-1}\big(\varphi_1(u)\big),\, a \circ \varphi_1'(u)^{-1} \circ \varphi_2'\big(\varphi_2^{-1}(\varphi_1(u))\big)\big)\\
&= \big((\varphi_2^{-1} \circ \varphi_1)(u),\, a \circ \varphi_1'(u)^{-1} \circ \varphi_2'\big((\varphi_2^{-1} \circ \varphi_1)(u)\big)\big)\\
&= \big((\varphi_2^{-1} \circ \varphi_1)(u),\, a \circ \varphi_1'(u)^{-1} \circ (\varphi_2^{-1})'\big(\varphi_1(u)\big)^{-1}\big)\\
&= \big((\varphi_2^{-1} \circ \varphi_1)(u),\, a \circ \big((\varphi_2^{-1})'\big(\varphi_1(u)\big) \circ \varphi_1'(u)\big)^{-1}\big)\\
&= \big((\varphi_2^{-1} \circ \varphi_1)(u),\, a \circ (\varphi_2^{-1} \circ \varphi_1)'(u)^{-1}\big),
\end{aligned}$$

where, as in the proof of (6.15), we used the fact that $(\varphi_2^{-1})'(x) = \varphi_2'\big(\varphi_2^{-1}(x)\big)^{-1}$ in the step from the third to the fourth line. This shows that if $\varphi_2^{-1} \circ \varphi_1$ is of class C^k then $\widehat{\varphi}_2^{-1} \circ \widehat{\varphi}_1$ is of class C^{k-1}. ■

Each C^k-mapping $f : M \to N$ between C^k-manifolds induces a C^{k-1}-mapping $f^\star : T^\star M \to T^\star N$ via $f^\star(p,\lambda) := \big(f(p), \lambda \circ f'(p)^{-1}\big)$. The chain rule shows that if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ then $(g \circ f)^\star = g^\star \circ f^\star$.

We conclude that this chapter with a theorem which allows in many cases to establish that a given set is, in fact, a manifold. We start with the following preliminary result.

(6.17) Rank Theorem. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets and let $f : U \to V$ be a mapping of class C^k such that $f'(x)$ has the constant rank r for all $x \in U$. Then, given a point $p \in U$, there exist an open neighborhood $U_0 \subseteq U$ of p, an open neighborhood $V_0 \subseteq V$ of $f(p)$ and C^k-diffeomorphisms $\varphi : U_0 \subseteq \varphi(U_0) \subseteq \mathbb{R}^m$ and $\psi : V_0 \to \psi(V_0) \subseteq \mathbb{R}^n$ such that*

$$(\psi \circ f \circ \varphi^{-1})(x_1, \ldots, x_m) \;=\; (x_1, \ldots, x_r, 0, \ldots, 0)$$

for all $x = (x_1, \ldots, x_m) \in \varphi(U_0)$. This means that f locally looks like the mapping $\mathbb{R}^m \to \mathbb{R}^r \times \{0\}$.)

Proof. Write $f(x) = \big(f_1(x), \ldots, f_n(x)\big)$. After renumbering coordinates if necessary, we may assume without loss of generality that $\det\big((\partial_j f_i)(x)_{i,j=1}^r\big) \neq 0$ in an open neighborhood $\widehat{U}$ of p. Define $\varphi : \widehat{U} \to \mathbb{R}^m$ by $\varphi(x_1, \ldots, x_m) := \big(f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m\big)$. Then

$$\varphi'(x) \;=\; \begin{bmatrix} (\partial_j f_i)(x)_{i,j=1}^r & \star \\ 0 & \mathbf{1} \end{bmatrix}$$

is invertible. The Inverse Function Theorem implies that there is an open neihborhood $U_0 \subseteq \widehat{U}$ of p such that $\varphi : U_0 \to \varphi(U_0)$ is a C^k-diffeomorphism. From the way φ is defined we see that $g := f \circ \varphi^{-1}$ has the form $g(\xi) = g(\xi_1, \ldots, \xi_m) = \big(\xi_1, \ldots, \xi_r, g_{r+1}(\xi), \ldots, g_n(\xi)\big)$ so that

$$(\star) \qquad g'(\xi) \;=\; \begin{bmatrix} \mathbf{1}_r & 0 \\ \star & A(\xi) \end{bmatrix}.$$

Up to this point we only used the fact that $\mathrm{rk} f'(x) \geq r$ in a neighborhood of p. Now we use the fact that the rank of $f'(x)$ (and hence that of $g'(\xi)$) is, in fact, equal to r; this fact implies that $A(\xi) = 0$ in $(\star)$, which means that $g_{r+1}(\xi), \ldots, g_n(\xi)$ do not depend on $\xi_{r+1}, \ldots, \xi_m$, but only on $\xi_1, \ldots, \xi_r$. Consequently,

$$\psi(\xi_1, \ldots, \xi_n) \;:=\; \big(\xi_1,\, \ldots,\, \xi_r,\, \xi_{r+1} - g_{r+1}(\xi_1, \ldots, \xi_r),\, \ldots,\, \xi_n - g_n(\xi_1, \ldots, \xi_r)\big)$$

is invertible, hence a C^k-diffeomorphism, and obviously $\psi \circ g = \psi \circ f \circ \varphi^{-1}$ has the desired form. ■

(6.18) Corollary. *Let $f : M \to N$ be a C^k-mapping between C^k-manifolds and let $y \in N$ be a given point. Assume that $f'(x)$ has constant rank r on an open neighborhood of $f^{-1}(y)$. Then $f^{-1}(y) = \{x \in M \mid f(x) = y\}$ is a C^k-manifold of dimension* $\dim(M) - r$.

Proof. By hypothesis, there are neighborhoods U of 0 in $\mathbb{R}^m$ and V of 0 in $\mathbb{R}^n$ and local parametrizations $\varphi : U \to M$ and $\psi : V \to N$ with $\varphi(0) = p$ and $\psi(0) = q$ such that $F := \psi^{-1} \circ f \circ \varphi$ satisfies $\mathrm{rk}\, F'(u) = r$ for all $u \in U$. Invoking Theorem (6.17), we may assume (after suitable coordinate changes and after making the neighborhoods U and V smaller if necessary) that F is in local coordinates given by $F(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$ so that $f^{-1}(q)$ cooresponds to $F^{-1}(0) = \{0\} \times \mathbb{R}^{m-r}$. Then φ maps $(\{0\} \times \mathbb{R}^{m-r}) \cap U$ diffeomorphically of class C^k onto $f^{-1}(q) \cap \varphi(U)$, which shows that $f^{-1}(q)$ sits locally inside M just as $\{0\} \times \mathbb{R}^{m-r}$ sits inside $\mathbb{R}^m$. This gives the claim. ■

Note that necessarily $r \leq \min\bigl(\dim(M), \dim(N)\bigr)$. The case that the rank is maximal is of particular importance.

(6.19) Definition. *Let $f : M \to N$ be a C^1-map between manifolds. We call f a* **submersion** *if $f'(x)$ is surjective for all $x \in M$ and an* **immersion** *if $f'(x)$ is injective for all $x \in M$. An immersion which is also a homeomorphism onto its image is called an* **embedding**.

(6.20) Example. We claim that $M := \{A \in \mathbb{R}^{n\times n} \mid A^T A = \mathbf{1}\}$ is a manifold of dimension $n(n-1)/2$. (We know this fact already, with M being nothing but the orthogonal group on $\mathbb{R}^n$, but we want to give a new proof of this fact.) Let Σ be the real vector space of all symmetric $(n \times n)$-matrices (which has dimension $n(n+1)/2$), and define $f : \mathbb{R}^{n\times n} \to \Sigma$ by $f(A) := A^T A$ so that $M = f^{-1}(\mathbf{1})$. Now

$$f'(A)X \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(A+tX) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (A^T + tX^T)(A+tX) \;=\; X^T A + A^T X.$$

We claim that if $A \in f^{-1}(\mathbf{1})$ (i.e., if $A^T A = \mathbf{1}$)) then $f'(A)$ is surjective. In fact, if $C \in \Sigma$ is an arbitrary symmetric matrix and if we let $X := (1/2)AC$ then $f'(A)X = (1/2)\bigl(C^T A^T A + A^T AC\bigr) = (1/2)\bigl(C^T + C\bigr) = C$. Consequently, $\mathrm{rk}\, F'(A) = n(n+1)/2$ for all $A \in f^{-1}(\mathbf{1})$, so that $f^{-1}(\mathbf{1})$ is a manifold of dimension $n^2 - n(n+1)/2 = (n^2 - n)/2$.

In Corollary (6.18), the set $f^{-1}(q)$ is a submanifold of M in the sense of the following definitin.

(6.21) Definition. *A subset $S \subseteq M$ of an m-dimensional C^k-manifold M is called a d-dimensional* **submanifold** *of M (more precisely: a d-dimensional* **embedded submanifold** *of M), if each point $p \in S$ possesses a local chart (U, ψ) (with respect to M) such that φ mapsto $S \cap U$ to $(\mathbb{R}^d \times \{0\}) \cap \psi(U)$ (i.e., if S sits locally in M just as $\mathbb{R}^d \times \{0\}$ sits in $\mathbb{R}^m$.*

If S is an abstractly given manifold and if $i : S \to M$ is an embedding into some manifold M then we sometimes identify S with $i(S)$, which is a submanifold of M. If $i : S \to M$ is only an immersion, then, by the Implicit Function Theorem, i is locally an embedding (which means that each point $s \in S$ possesses a neighborhood S_0 such that $i|_{S_0}$ is an embedding). Now if i is an injective immersion, some authors call $i(S)$ an *immersed submanifold*, but this is somewhat misleading, as $i(S)$ need not be a submanifold in the sense of Definition (6.21).

(6.22) Examples. (a) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) := 2\arctan\bigl(t - \arctan(\pi/2)\bigr) + (\pi/2)$ and $i : \mathbb{R} \to \mathbb{R}^2$ by $i(t) := \Bigl(2\cos\bigl(g(t)\bigr), \sin\bigl(2g(t)\bigr)\Bigr)$. Then i is an injective immersnion, but not an embedding. (The image of i looks like a figure eight in which no neighborhood of $(0,0)$ is homeomorphic to an open interval $I \subseteq \mathbb{R}$. The image of i is, however, a manifold diffeomorphic to $\mathbb{R}$ if we do not equip it with the subspace topology from $\mathbb{R}^2$ but with an intrinsic topology which makes it homeomorphic to the real line.)

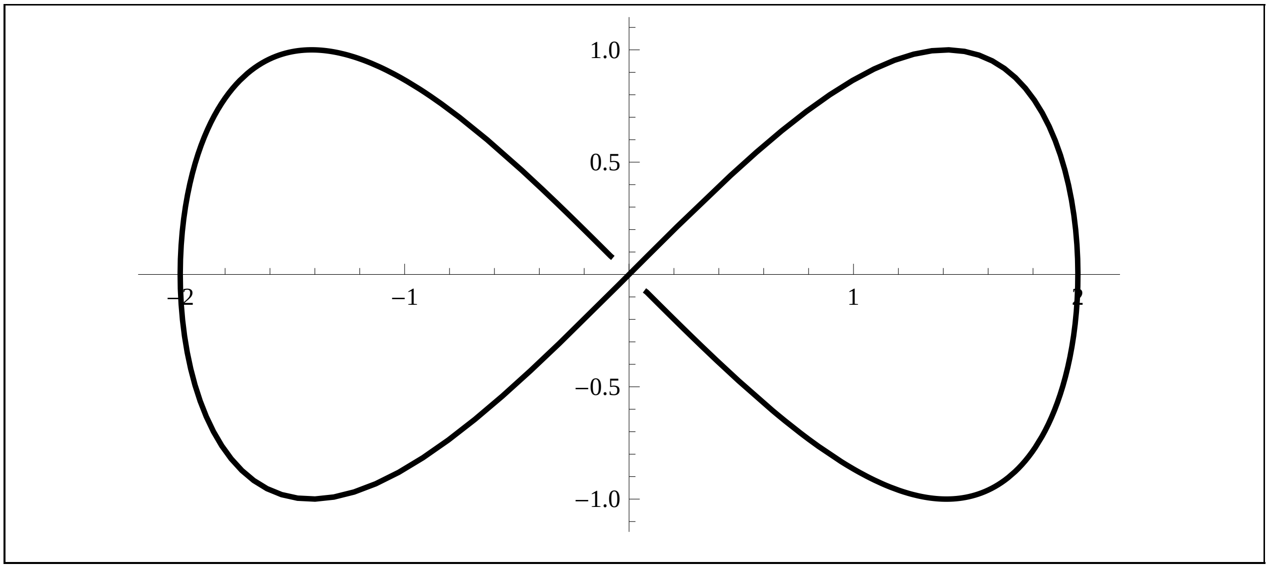

Fig. 6.5: Immersed submanifold which is not an embedded submanifold.

(b) Consider the mapping $i : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $i(t) := \big(e^{it}, e^{i\alpha t}\big)$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number. Then i is an injective immersion, but not an embedding, because the image of i is dense in the torus $\mathbb{S}^1 \times \mathbb{S}^1$ so that the intrinsic topology of the image of i (when considered as a one-dimensional manifold diffeomorphic to $\mathbb{R}$) does not coincide with the subspace topology inherited from $\mathbb{S}^1 \times \mathbb{S}^1$.

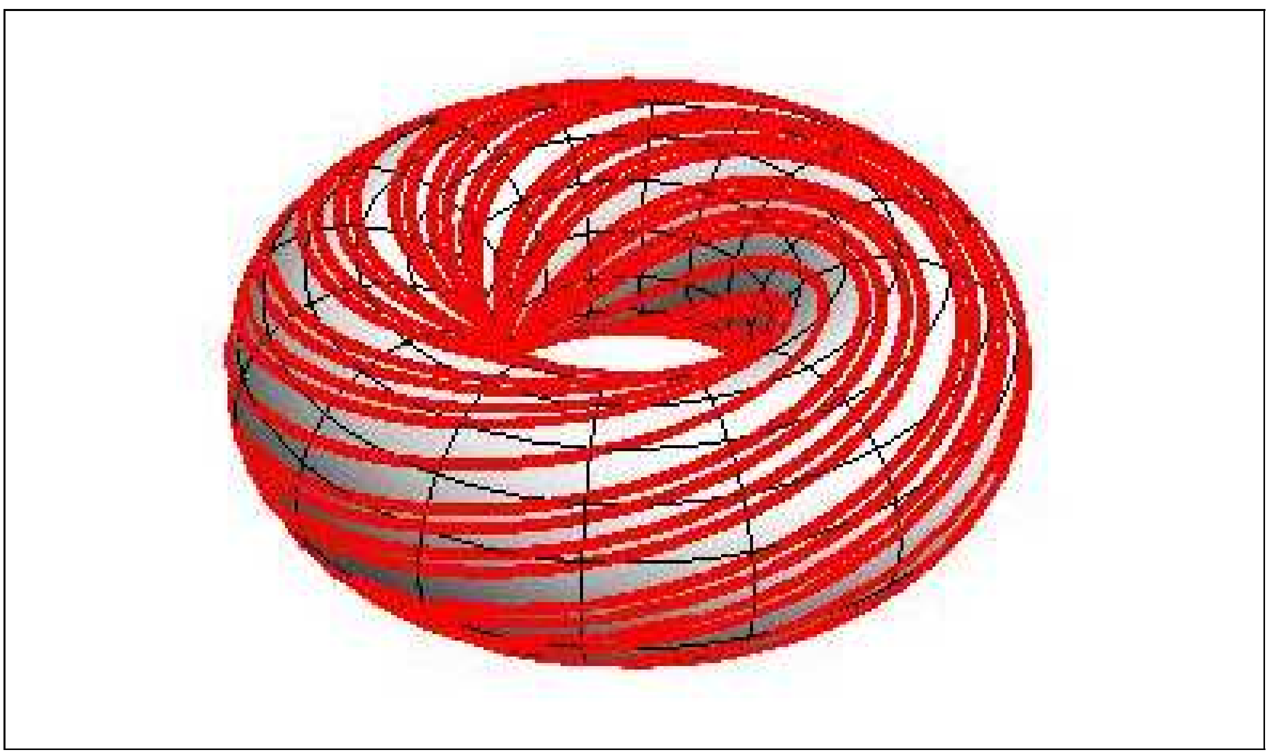

Fig. 6.6: Dense curve winding around a torus.