3. Tangent and Cotangent Spaces

In this section we introduce for each manifold $M \subseteq \mathbb{R}^n$ and each point $p \in M$ a "linear approximation" of M around p, namely the *tangent space* of M at p. This will, in arbitrary dimensions, be the analogue of the tangent line of a curve at a point or the tangent plane of a surface at a point.

(3.1) Definition. *Let $M \subseteq \mathbb{R}^n$ be a manifold in $\mathbb{R}^n$ and let $p \in M$ be a point in M. The* **tangent space** *of M at p is the set T_pM of all vectors $v \in \mathbb{R}^n$ with the following property: There are an open interval $I \subseteq \mathbb{R}$ and a C^1-curve $\alpha : I \to \mathbb{R}^n$ with $\alpha(t) \in M$ for all $t \in I$ such that $\alpha(t_0) = p$ and $\alpha'(t_0) = v$ for some $t_0 \in I$.*

In other words: the tangent space T_pM of M at p is the set of all velocity vectors with which smooth curves lying entirely in M can pass through the point p. We immediately make the following observations.

- The definition of T_pM makes sense for an arbitrary subset $M \subseteq \mathbb{R}^n$. However, to obtain meaningful results it will be necessary to assume that M is indeed a manifold of class C^1. (Nothing is gained if M is a manifold of class C^k where $k > 1$.)
- To calculate $\alpha(t_0)$ and $\alpha'(t_0)$, only the values of α in an arbitrarily small interval around t_0 are needed; hence it is enough to consider curves defined on some interval $(t_0 - \varepsilon,\, t_0 + \varepsilon)$. Moreover, we may always assume $t_0 = 0$ by replacing α by the curve $t \mapsto \alpha(t + t_0)$ (which is defined in an open interval containing 0).
- The very definition of a tangent space shows that if x is a point of a manifold $M \subseteq \mathbb{R}^n$ and if $U \subseteq \mathbb{R}^n$ is an open neighborhood of x, then $T_xM = T_x(M \cap U)$. The tangent space is a purely local object and does not depend on the global properties of the manifold M.
- Given a point $p \in M$ and a tangent vector $v \in T_pM$, there is a curve $t \mapsto \alpha(t)$ in M with $\alpha(0) = p$ and $\alpha'(0) = v$. Given a real number r, we can consider the curve $\beta(t) := \alpha(rt)$ $\big($defined on $(-\varepsilon/r,\, \varepsilon/r)$ if α is defined on $(-\varepsilon, \varepsilon)\big)$ which traces out the same path as α (and hence also lies entirely in M), but with different speed. More precisely, we have satisfies $\beta(0) = \alpha(0) = p$ and $\beta'(0) = r\alpha'(0) = rv$; hence rv is also a tangent vector of M at p. This shows that scalar multiples of tangent vectors are again tangent vectors.
- It turns out that the sum of two tangent vectors is again a tangent vector. (This will be an immediate consequence of the followign theorem.) Thus the tangent space T_pM is indeed a vector space (which justifies the designation "space").
- The tangent space T_pM is a vector subspace of $\mathbb{R}^n$ and hence contains the zero vector $0 \in \mathbb{R}^n$. However, for visualization purposes it is easier to think of the affine space $p + T_pM$ which is obtained by translating T_pM through the "base point" p. Thus we often think of T_pM as being attached to the point p.

We now want to show how the tangent space of a concrete manifold can be determined. Since there are essentially two ways to represent a manifold (namely by parametrizations and by equations), there are also two ways to describe the tangent spaces of a manifold. We start with manifolds given in terms of a parametrization.

(3.2) Theorem. *Assume that M is locally given by*

$$M \cap U \;=\; \{\varphi(\xi) \mid \xi \in \Omega\}$$

for some C^1-parametrization $\varphi : \Omega \to \mathbb{R}^n$ with $\Omega \subseteq \mathbb{R}^d$. Then the tangent space of M at some point $p = \varphi(\xi_0)$ is given by

$$T_pM \;=\; \{\varphi'(\xi_0)w \mid w \in \mathbb{R}^d\} \;=\; \operatorname{im} \varphi'(\xi_0)\,.$$

Proof. Given any C^1-curve $t \mapsto \xi(t)$ in Ω with $\xi(0) = \xi_0$ yields a C^1-curve $\alpha(t) := \varphi\big(\xi(t)\big)$ lying in M and satisfying $\alpha(0) = p$. Conversely, every C^1-curve $\alpha : I \to \mathbb{R}^n$ lying entirely in M and satisfying $\alpha(0) = p$ is of this form (at least on a small interval around 0). Since for such a curve we have $\alpha'(t) = \varphi'\big(\xi(t)\big)\xi'(t)$ and hence $\alpha'(0) = \varphi'(\xi_0)\xi'(0)$, we see that T_pM consists of all vectors of the form $\varphi'(\xi_0)w$ where $w \in \mathbb{R}^d$ is a tangent vector to a curve in Ω at ξ_0. Since *every* vector $w \in \mathbb{R}^d$ arises as the tangent vector of a suitably chosen curve in Ω (for example, we can choose $\xi(t) = \xi_0 + tw$), this gives the claim. ■

Thus if M is locally the image of the parametrization φ, the tangent space of M at $\varphi(\xi_0)$ is the image of the linearization $\varphi'(\xi_0)$ of φ at ξ_0: The linearization of M is obtained by linearizing the parametrization. Since $\varphi'(\xi_0) \in \mathbb{R}^{n\times d}$ has rank d, the image of $\varphi'(\xi_0)$ is a d-dimensional vector subspace of $\mathbb{R}^n$. Thus Theorem (3.2) immediately implies the following result.

(3.3) Corollary. *If $M \subseteq \mathbb{R}^n$ is a d-dimensional manifold and if $p \in M$ is a point in M, then T_pM is a d-dimensional vector subspace of $\mathbb{R}^n$.*

Similarly, we can find a dual description of the tangent space if the manifold in question is given as the solution set of a system of equations.

(3.4) Theorem. *Assume that M is locally given by*

$$M \cap U \;=\; \{x \in U \mid g(x) = 0\}$$

where $U \subseteq \mathbb{R}^n$ is open and $g : U \to \mathbb{R}^{n-d}$ is a C^1-function of constant rank $n-d$. Then the tangent space of M at a point $p \in M \cap U$ is given by

$$T_pM \;=\; \{v \in \mathbb{R}^n \mid g'(p)v = 0\} \;=\; \ker g'(p)\,.$$

Proof. Let $v \in T_pM$. Then there is a C^1-curve $\alpha : I \to U$ lying in M such that $\alpha(0) = p$ and $\alpha'(0) = v$. Since α lies entirely in M we have $g\big(\alpha(t)\big) = 0$ for all $t \in I$. Hence the derivative $g'\big(\alpha(t)\big)\alpha'(t) \equiv 0$ is also identically zero. Especially for $t = 0$ we find that $g'(p)v = 0$ and hence $v \in \ker g'(p)$. Since $v \in T_pM$ was arbitrary, we have shown that

$$(\star) \qquad T_pM \;\subseteq\; \ker g'(p)\,.$$

Now the left-hand side of $(\star)$ is a vector space of dimension d (as we know from Theorem (3.2)). The right-hand side of $(\star)$ is a vector space of dimension $n - \dim \operatorname{im} g'(p) = n - \operatorname{rk} g'(p) = n - (n-d) = d$. This shows that the inclusion $(\star)$ is, in fact, an equality. ■

Theorem (3.4) says that the linearization of the zero set of g at p is given as the zero set of the linearization $g'(p)$ of g at p. Thus the two dual ways of defining a manifold (by general nonlinear equations and parametrizations) yield, after linearizing, the two dual ways to define the tangent space (by affine equations and parametrizations).

(3.5) Example. Let us determine the tangent space (i.e., the tangent plane) to the surface $M = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^3\}$ at the point $p = (1,1,2)$ (which obviously lies in M). Since M is given as the zero set of the function $g(x,y,z) := x^2 + y^3 - z$, whose derivative is $g'(x,y,z) = (2x, 3y^2, -1)$, the tangent space $T_pM = \ker g'(p)$ is given by

$$(1) \qquad T_pM \;=\; \ker g'(1,1,1) \;=\; \ker\,(2,3,-1) \;=\; \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + 3y - z = 0 \right\}.$$

On the other hand, we can use the parametrization

$$\varphi(u,v) = \begin{bmatrix} u \\ v \\ u^2 + v^3 \end{bmatrix} \quad \text{with} \quad \varphi'(u,v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 3v^2 \end{bmatrix}$$

which maps $\xi_0 := (u_0, v_0) := (1,1)$ to p. Therefore, the tangent space $T_pM = \operatorname{im} \varphi'(\xi_0)$ is given by

$$(2) \qquad T_pM \;=\; \operatorname{im} \varphi'(1,1) \;=\; \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \;=\; \left\{ u \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \;\middle|\; u, v \in \mathbb{R} \right\}.$$

Note that (1) yields the plane T_pM in normal form whereas (2) yields T_pM in parameter form. ■

(3.6) Example. The unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is the zero set of the function $g(x_1, \ldots, x_n) := x_1^2 + \cdots + x_n^2 - 1$. Since $g'(x) = (2x_1, \ldots, 2x_n)$, we have $T_x\mathbb{S}^{n-1} = \{v \in \mathbb{R}^n \mid g'(x)v = 0\} = \{v \in \mathbb{R}^n \mid 2(x_1v_1 + \cdots + x_nv_n) = 0\}$, i.e.,

$$T_x(\mathbb{S}^{n-1}) \;=\; \{v \in \mathbb{R}^n \mid \langle v, x\rangle = 0\} \;=\; \{v \in \mathbb{R}^n \mid v \perp x\}$$

where $\langle \cdot, \cdot \rangle$ is the ordinary inner product on $\mathbb{R}^n$. In dimensions 2 and 3 this is obvious geometrically.

The tangent spaces of a manifold M at its various points can be combined to a set as follows.

(3.7) Definition. *Let $M \subseteq \mathbb{R}^n$ be a manifold. Then the* **tangent bundle** TM *of* M *is defined as*

$$TM \;:=\; \{(p,v)\in\mathbb{R}^n\times\mathbb{R}^n \mid p\in M,\ v\in T_pM\}.$$

The mapping $TM \to M$ given by $(p,v)\mapsto p$ is called the **basepoint projection**.

As a set, TM can be simply considered as the disjoint union $\bigcup_{p\in M} T_pM$ of all the tangent spaces of M at its various points. However, TM is not just a set but a manifold in its own right.

(3.8) Theorem. *If $M \subseteq \mathbb{R}^n$ is a C^k-manifold of dimension d where $k>1$, then $TM \subseteq \mathbb{R}^n\times\mathbb{R}^n = \mathbb{R}^{2n}$ is a C^{k-1}-manifold of dimension $2d$.*

Proof. We establish the claim in two different ways. Let us first assume that $\varphi:\Omega\to\mathbb{R}^n$ is a local parametrization of M where $\Omega\subseteq\mathbb{R}^d$. Then $\bigcup_{p\in\varphi(\Omega)} T_pM = \{\big(\varphi(\xi),\varphi'(\xi)w\big) \mid \xi\in\Omega, w\in\mathbb{R}^d\}$ is an open subset of TM which is parametrized by the C^{k-1}-mapping

$$\Phi:\Omega\times\mathbb{R}^d\to\mathbb{R}^n\times\mathbb{R}^n,\quad \Phi(\xi,w) := \begin{bmatrix}\varphi(\xi)\\ \varphi'(\xi)w\end{bmatrix}.$$

This is indeed a regular parametrization because

$$\Phi'(\xi,w) \;=\; \begin{bmatrix}\varphi'(\xi) & 0\\ \varphi''(\xi)w & \varphi'(\xi)\end{bmatrix}$$

is a matrix of rank $2d$ (because $\varphi'(\xi)$ has rank d).

Let us now assume that M is locally given by $M\cap U = \{x\in U \mid g(x)=0\}$ for some open set $U\subseteq\mathbb{R}^n$ and some C^k-mapping $g:U\to\mathbb{R}^{n-d}$. Then $\bigcup_{p\in M\cap U} T_pM$ is the set $\{(x,v)\in U\times\mathbb{R}^n \mid g(x)=0, g'(x)v=0\}$ and hence the zero set of the mapping

$$G:U\times\mathbb{R}^n\to\mathbb{R}^{n-d}\times\mathbb{R}^{n-d},\qquad G(x,v) \;=\; \begin{bmatrix}g(x)\\ g'(x)v\end{bmatrix}.$$

Note that if g is of class C^k then G is of class C^{k-1}. Moreover,

$$G'(x,v) \;=\; \begin{bmatrix}g'(x) & 0\\ g''(x)v & g'(x)\end{bmatrix}$$

has rank $(n-d)+(n-d)=2n-2d$, because $g'(x)$ has rank d. ■

(3.9) Interpretation. If M is the *configuration space* of a physical system, for example the set of all positions a system of rigid bodies or point masses can occupy, then TM consists of all pairs (p,v) where p is a possible configuration of the system and v is a possible direction in which this configuration can change. Thus TM encodes all possible configurations of the system along with the possible tendencies of the system to change its current configuration. In this situation the tangent bundle TM is called the *phase space* of the system in question.

(3.10) Example. If $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, we have

$$T\mathbb{S}^{n-1} = \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|x\| = 1, \langle x, v\rangle = 0\},$$

which is the zero set of the C^∞-function

$$G(x,v) := \begin{bmatrix} \|x\|^2 - 1 \\ \langle x, v\rangle \end{bmatrix} \quad \text{with} \quad G'(x,v) = \begin{bmatrix} 2x^T & 0 \\ v^T & x^T \end{bmatrix}.$$

■

There is a dual concept to that of a tangent space, which is more algebraic than geometric in nature and hence less easily understood intuitively, but turns out to be just as important as the concept of tangent space.

(3.11) Definition. *Let $M \subseteq \mathbb{R}^n$ be a manifold in $\mathbb{R}^n$ and let $p \in M$ be a point in M. The* **cotangent space** *of M at p is the set $(T_pM)^\star$ of all linear forms on T_pM, i.e., of all linear mappings $f : T_pM \to \mathbb{R}$.*

Note that since T_pM is a vector subspace of $\mathbb{R}^n$, a linear form on T_pM is the restriction of a linear form on $\mathbb{R}^n$, and each linear form of $\mathbb{R}^n$ is given by a row vector $a \in (\mathbb{R}^n)^\star$. Now if M is given by a parametrization φ, then the tangent space $T_{\varphi(\xi)}M$ is given by $\operatorname{im}\varphi'(\xi)$. Thus the linear forms on $T_{\varphi(\xi)}M$ can be identified with the linear forms on $\mathbb{R}^n$ vanishing on $\big(\operatorname{im}\varphi'(\xi)\big)^\perp$, which gives

$$\begin{aligned}
(T_{\varphi(\xi)}M)^\star &= \{a \in (\mathbb{R}^n)^\star \mid a \equiv 0 \text{ on } \big(\operatorname{im}\varphi'(\xi)\big)^\perp\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a^T \in \big(\operatorname{im}\varphi'(\xi)\big)^{\perp\perp}\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a^T \in \operatorname{im}\varphi'(\xi)\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a^T = \varphi'(\xi)w \text{ for some } w \in \mathbb{R}^d\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a = w^T\varphi'(\xi)^T \text{ for some } w \in \mathbb{R}^d\} \\
&= \{w^T\varphi'(\xi)^T \mid w \in \mathbb{R}^d\} \\
&= \{b^T \mid b \in \operatorname{im}\varphi'(\xi)\} \\
&= \{b^T \mid b \in T_{\varphi(\xi)}M\}\,.
\end{aligned}$$

Similarly, if M is locally given as the zero set of a mapping $g : \mathbb{R}^n \to \mathbb{R}^{n-d}$ so that $T_pM = \ker g'(p)$, we can identify $(T_pM)^\star$ with the space of all linear forms on $\mathbb{R}^n$ which vanish on $\big(\ker g'(p)\big)^\perp = \operatorname{im} g'(p)^T$. This yields

$$\begin{aligned}
(T_pM)^\star &= \{a \in (\mathbb{R}^n)^\star \mid a \equiv 0 \text{ on } \operatorname{im} g'(p)^T\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a\, g'(p)^T = 0\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a^T \in \ker g'(p)\} \\
&= \{a \in (\mathbb{R}^n)^\star \mid a^T \in T_pM\}\,.
\end{aligned}$$

(3.12) Definition. *Let $M \subseteq \mathbb{R}^n$ be a manifold. Then the* **cotangent bundle** $T^\star M$ *of M is defined as*

$$T^\star M \;:=\; \{(p,f)\in\mathbb{R}^n\times\mathbb{R}^n \mid p\in M,\ f\in(T_pM)^\star\}.$$

The mapping $T^\star M\to M$ given by $(p,f)\mapsto p$ is called the **basepoint projection***.*

As a set, $T^\star M$ can be simply considered as the disjoint union $\bigcup_{p\in M}(T_pM)^\star$ of all the cotangent spaces of M at its various points. However, as was already the case with the tangent bundle TM, the cotangent bundle carries a manifold structure in a natural way.

(3.13) Theorem. *If $M\subseteq\mathbb{R}^n$ is a C^k-manifold of dimension d where $k>1$, then $T^\star M\subseteq\mathbb{R}^n\times\mathbb{R}^n=\mathbb{R}^{2n}$ is a C^{k-1}-manifold of dimension $2d$.*

Proof. The result follows immediately from the calculations preceding (3.12). If M is locally parametrized by a C^k-mapping $\varphi:\Omega\to\mathbb{R}^n$ where $\Omega\subseteq\mathbb{R}^d$ then $T^\star M$ is locally parametrized by the C^{k-1}-mapping

$$\Phi:\Omega\times\mathbb{R}^d\to\mathbb{R}^n\times\mathbb{R}^n,\qquad \Phi(\xi,w)\;:=\;\big(\varphi(\xi),w^T\varphi'(\xi)^T\big)\,.$$

If M is locally given as the zero set of a C^k-mapping $g:U\to\mathbb{R}^{n-d}$ where $U\subseteq\mathbb{R}^n$ then $T^\star M$ is locally the zero set of the C^{k-1}-mapping

$$G:U\times\mathbb{R}^n\to\mathbb{R}^{n-d}\times\mathbb{R}^{n-d},\qquad G(x,v):=\big(g(x),v^Tg'(x)^T\big).$$

■

(3.14) Definition. *Let $M_1,M_2\subseteq\mathbb{R}^n$ be manifolds with dimensions $d_1=\dim(M_1)$ and $d_2=\dim(M_2)$ such that $d_1\le d_2$. We say that M_1* **touches** *M_2 or is* **tangent** *to M_2 at a point $p\in\mathbb{R}^n$ if $p\in M_1\cap M_2$ and $T_pM_1\subseteq T_pM_2$.*

Note that if $d_1=d_2$ the condition $T_pM_1\subseteq T_pM_2$ becomes $T_pM_1=T_pM_2$. We now define an *envelope* of a family of hypersurfaces to be a manifold which touches each of these hypersurfaces at a unique point. (In concrete examples singular points may occur, or the envelope may touch one of the hypersurfaces in more than one place. In these cases we tacitly assume that singular points are removed and considerations are restricted to smaller regions in which all the hypotheses are satisfied.)

(3.15) Definition. *Let (M_c) be a family of hypersurfaces in $\mathbb{R}^n$, parametrized by a parameter $c=(c_1,\ldots,c_k)\in\mathbb{R}^k$. We call a manifold M an* **envelope** *of the family (M_c) if M shares with each manifold M_c a unique point $x(c)$ at which M is tangent to M_c, which means that $T_{x(c)}M\subseteq T_{x(c)}M_c$ for all parameter values c.*

(3.16) Examples. (a) For $c\in\mathbb{R}$ let $M_c:=\{(x,y)\in\mathbb{R}^2\mid y=(x-c)^2\}$. This is a family of parabolas, and it is obvious that the x-axis $M:=\{(x,y)\in\mathbb{R}^2\mid y=0\}$ is an envelope of the family (M_c).

(b) Let $r > 0$ be a fixed number. For $c \in \mathbb{R}$, let $M_c := \{(x,y,z) \in \mathbb{R}^3 \mid (x-c)^2 + y^2 + z^2 = r^2\}$. This is a family of spheres of radius r with centres along the x-axis. It is obvious that each of the two lines $\{(x, 0, \pm r) \mid x \in \mathbb{R}\}$ is an envelope of the family (M_c).

(c) Let $r > 0$ be a fixed number. For $c = (c_1, c_2) \in \mathbb{R}^2$, let $M_c := \{(x,y,z) \in \mathbb{R}^3 \mid (x-c_1)^2 + (y-c_2)^2 + z^2 = r^2\}$. This is a family of spheres of radius r with centres along the xy-plane. It is obvious that each of the two planes $\{(x, y, \pm r) \mid x, y \in \mathbb{R}\}$ is an envelope of the family (M_c).

We now show how an envelope can be found for a given family of hypersurfaces (which we assume to be available in equation form).

(3.17) Theorem. *Let M be an envelope of a family (M_c) of hypersurfaces $M_c \subseteq \mathbb{R}^n$ and assume that $M_c = \{x \in \Omega \mid g(x,c) = 0\}$ for some C^1-function g defined on some open subset $\Omega \subseteq \mathbb{R}^n$. We assume that the parameters $c = (c_1, \ldots, c_k)$ range over some open parameter domain $B \subseteq \mathbb{R}^k$ and that the contact point $x(c)$ between M and M_c depends smoothly on the parameter c. Then*

$$(\partial_c g)\big(x(c), c\big) \;=\; 0.$$

Proof. Since each point $x(c)$ belongs to M_c, the identity $g\big(x(c), c\big) = 0$ holds. Taking in this equation the derivative with respect to the parameter c, we find that

$$(\star) \qquad 0 \;=\; (\partial_x g)\big(x(c), c\big)\, x'(c) + (\partial_c g)\big(x(c), c\big).$$

On the other hand, each column of $x'(c)$ is a tangent vector to M, because if c_j is kept constant for $j \neq i$, the mapping $c_i \mapsto x(c_1, \ldots, c_i, \ldots, c_k)$ parametrizes a curve in M whose tangent vector $\partial x/\partial c_i$ is just the i-th column of $x'(c)$. This means that $\partial x/\partial c_i$ lies in $T_{x(c)}M = T_{x(c)}M_c = \ker\big((\partial_x g)\big(x(c), c\big)\big)$ so that $(\partial_x g)\big(x(c), c\big)(\partial x/\partial c_i) = 0$. This implies that $(\partial_x g)\big(x(c), c\big)x'(c) = 0$, so that the first term on the right-hand side of $(\star)$ vanishes. The claim follows. ∎

(3.18) Remark. Each point $x = x(c)$ of M satisfies the two equations $g(x,c) = 0$ and $(\partial_c g)(x,c) = 0$. Elimination of c from these equations results in an equation $h(x) = 0$, which then is the equation describing the envelope. In the special case $n = 2$ and $k = 1$, we have two equations $g(x_1, x_2, c) = 0$ and $(\partial_c g)(x_1, x_2, c) = 0$. In this case one can also solve for x_1 and x_2 in terms of c, thereby obtaining a parametrization $c \mapsto \big(x_1(c), x_2(c)\big)$ for the envelope (which is a plane curve in this case).

(3.19) Example. The differential equation $y = xy' - (y')^2$ has the obvious solutions $y(x) = cx - c^2$, which constitute a one-parameter family $\mathfrak{F}$ of straight lines in $\mathbb{R}^2$. To see whether or not this family possesses an envelope, we take the derivative with respect to c to obtain $0 = x - 2c$, resulting in $c = x/2$. Plugging this back into the equation $y = cx - c^2$ yields $y(x) = x^2/4$. Thus the parabola with the equation $y = x^2/4$ is an envelope of the family $\mathfrak{F}$. The function $y(x) = x^2/4$ is again a solution of the differential equation $y = xy' - (y')^2$.

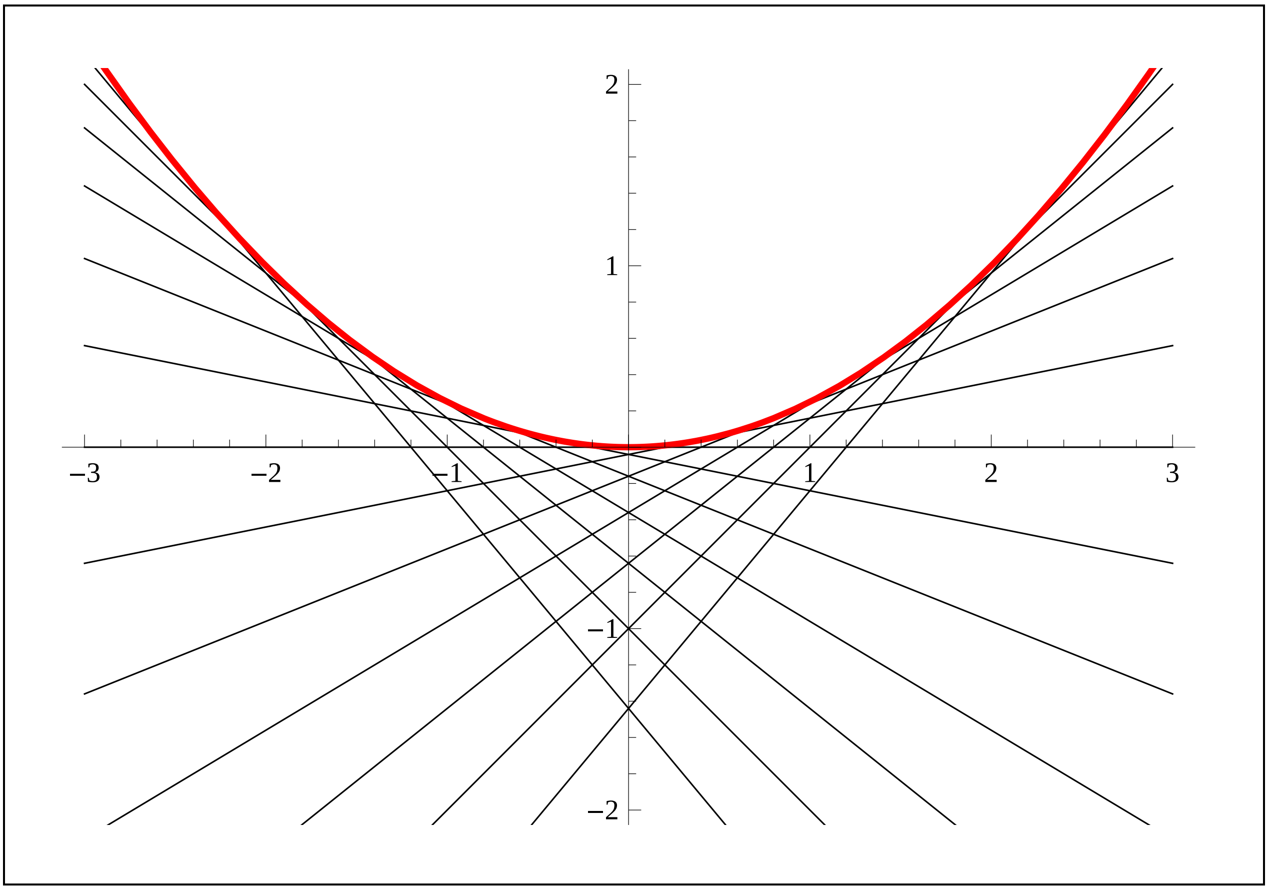

Fig. 3.1: Parabola $y = x^2/4$ as the emvelope of the lines $y = cx - c^2$ where $c \in \mathbb{R}$.