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Option Pricing:

The Main Idea

Main Idea of Option Pricing

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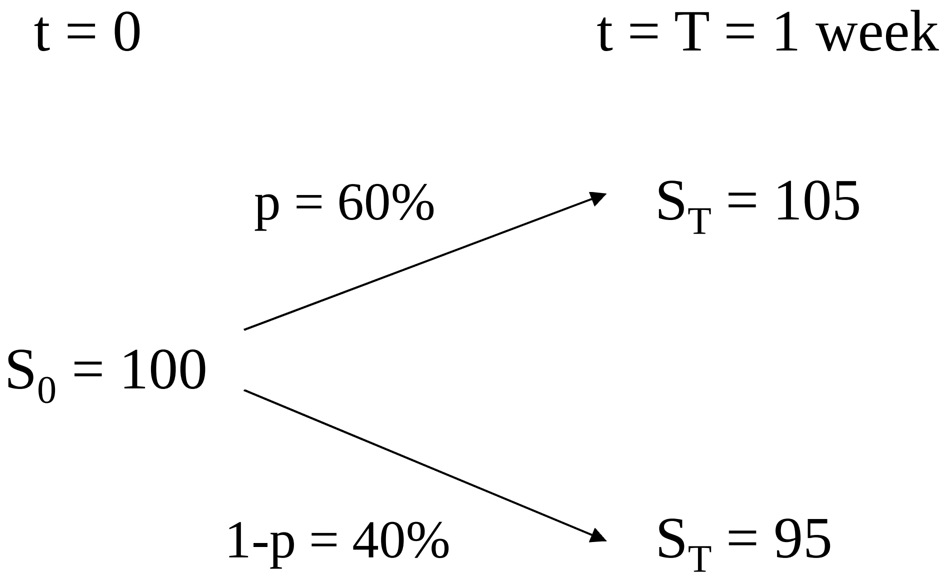
- What are options or derivatives ?

- Pure mathematically speaking: An option with underlying S_t (could be a stock) with maturity T is some arbitrary payoff - function $f(S_T)$.

  At maturity $t = T$, the buyer of the option f receives the amount $f(S_T)$ from the option seller.

A Simple Example:

- Consider some stock with a time horizon of 1 week:



A Simple Example:

- Standard-Call-Option: $f(S_T) := \max\{ S_T - S_0 , 0 \}$

$t = 0$ | $t = T = 1$ week | Option Payoff:

$p = 60\%$ → $S_T = 105$ | $5 = \max\{ 105 - 100 , 0 \}$

$S_0 = 100$

$1\text{-}p = 40\%$ → $S_T = 95$ | $0 = \max\{ 95 - 100 , 0 \}$

A Simple Example:

- Standard-Call-Option: $f(S_T) := \max\{\, S_T - S_0 \,,\, 0 \,\}$

t = 0   t = T = 1 week   Option Payoff:

p = 60% → $S_T = 105$ $5 = \max\{\, 105 - 100 \,,\, 0 \,\}$

$S_0 = 100$

1-p = 40% → $S_T = 95$ $0 = \max\{\, 95 - 100 \,,\, 0 \,\}$

- QUESTION: How much would you pay for this at $t = 0$?

A Simple Example:

- One may think:

  option_price = 60% * 5 Euro + 40% * 0 Euro = 3 Euro .

- This is wrong as we will see in a minute. Suppose it would be right. The following could happen:

  A large investor buys 1 million of these options at some bank. Thus, the bank receives at t = 0 the amount of 3 millionen Euros.

  The time of 1 week goes by and the stock is either up to 105 or down to 95.

A Simple Example:

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- If the stock is down, the bank does not need to pay anything to the investor and makes an instant profit of 3 million.
- If the stock is up, the bank has to pay 5 million to the investor and makes an instant loss of 2 million.

- Banks cannot take these kind of risks. Instead, like a car dealer, a bank wants to make a profit of a couple percent per sold option, irregardless of the fact whether the stock is going up or down.

- Now, it is a fundamental resultat of mathematical finance, that this is indeed possible. To this end, the bank has to do the following:

A Simple Example:

- The bank has to charge only 2,50 Euro as the option price (not 3 Euro).
- Then, at $t = 0$, the bank has to buy 0.5 stocks.
- At option maturity $t = T = 1$ week, the bank resells these 0.5 stocks:

BankPortfolio_today = 2,50 = 2,50 – 50 + 50
= -47,50(cash) + half a stock

BankPortfolio_1week = -47,50(cash) +
- 105/2 = -47,50+52,50 = 5 Euro
- 95/2 = -47,50+47,50 = 0 Euro

Thus: **BankPortfolio_1week = option_payoff**

A Simple Example:

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- Thus: With the trading strategy "buy half a stock" the bank is able to exactly replicate the option payoff.
- Then, the **fair price of an option** is the money which is needed to set up a replicating strategy.
- In this example this was just **2,50 Euro** .
- The actual traded price of that option is then probably 2,55 Euro or 2,53 Euro oder 2,52 Euro...

- Wenn man diese Idee in einem entsprechenden mathematischen Modell formalisiert, was tatsaechlich von Banken angewendet werden kann, sieht das folgendermassen aus (the most sophisticated version with stochastic volatility and stochastic interest rates, Kapitel 18 Finanzmathematik II):

A Realistic Model with Stoch Vol and Stochastic Rates:

Theorem: Suppose that the real world (not risk neutral) processes for some stock S_t, variance ν_t and short term interest rate r_t are given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu\,dt + \sqrt{\nu_t}\,dB_t^S \\ d\nu_t &= \alpha(\nu_t,t)\,dt + \beta\sqrt{\nu_t}\,dB_t^\nu \\ dr_t &= m(r_t,t)\,dt + \sigma\,dB_t^r \end{aligned} \tag{52}$$

with correlations

$$\begin{aligned} dB^S \cdot dB^\nu &= \rho_{S,\nu}\,dt \\ dB^S \cdot dB^r &= \rho_{S,r}\,dt \\ dB^\nu \cdot dB^r &= \rho_{\nu,r}\,dt \end{aligned} \tag{53}$$

Let $H = H(S_T)$ be the payoff of some derivative which is to be priced and let

$$v_t = v_0 + \int_0^t \delta_u\,ds_u + \int_0^t \eta_u\,dc_u + \int_0^t \rho_u\,dp_u \tag{54}$$

be the discounted time t value of a self financing strategy which holds at time u δ_u stocks, η_u plain vanilla options $C(S_u,\nu_u,r_u,u)$ and ρ_u zero bonds $P(r_u,u)$. Then the following statements hold:

A Realistic Model with Stoch Vol and Stochastic Rates:

a) If the hedge instruments P and C are consistently priced in the model (52), then the functions

$$\begin{aligned} \tilde{m}(r,t) &:= -\frac{\frac{\sigma^2}{2}P_{rr}+P_t-rP}{P_r} && (55) \\ \tilde{\alpha}(S,\nu,r,t) &:= -\frac{rSC_S+\tilde{m}C_r+\frac{1}{2}(S^2\nu C_{SS}+\beta^2\nu C_{\nu\nu}+\sigma^2 C_{rr})}{C_\nu} \\ &\quad -\frac{\beta\nu S\rho_{S,\nu}C_{S\nu}+\sqrt{\nu}\sigma S\rho_{S,r}C_{Sr}+\beta\sqrt{\nu}\sigma\rho_{\nu,r}C_{\nu r}+C_t-rC}{C_\nu} && (56) \end{aligned}$$

have to be some universal functions independent of the particular choice of P and C.

b) Define the differential operator (for some function $V = V(S,\nu,r,t)$)

$$\begin{aligned} \mathcal{L}_{\text{risk neutral}}V &:= rSV_S+\tilde{\alpha}V_\nu+\tilde{m}V_r+\tfrac{1}{2}\big(S^2\nu V_{SS}+\beta^2\nu V_{\nu\nu}+\sigma^2 V_{rr}\big) && (57) \\ &\quad +\beta\nu S\rho_{S,\nu}V_{S\nu}+\sqrt{\nu}\sigma S\rho_{S,r}V_{Sr}+\beta\sqrt{\nu}\sigma\rho_{\nu,r}V_{\nu r}+V_t-rV \end{aligned}$$

Suppose that V is a solution of the PDE

$$\begin{aligned} \mathcal{L}_{\text{risk neutral}}V &= 0 && (58) \\ V(S,\nu,r,T) &= H(S_T) \end{aligned}$$

A Realistic Model with Stoch Vol and Stochastic Rates:

and define

$$\begin{aligned} \eta &:= \frac{V_\nu}{C_\nu} && (59)\\ \delta &:= V_S - \eta C_S && (60)\\ \rho &:= \frac{V_r}{P_r} - \eta \frac{C_r}{P_r} && (61) \end{aligned}$$

Then (54) is in fact a replicating strategy for H. That is,

$$V(S,\nu,r,0) + \int_0^T \delta_t \, ds_t + \int_0^T \eta_t \, dc_t + \int_0^T \rho_t \, dp_t = e^{-\int_0^T r_t dt} H(S_T) \qquad (62)$$

where

$$\begin{aligned} s_t &= e^{-\int_0^T r_t dt} S_t \\ c_t &= e^{-\int_0^T r_t dt} C(S_t, \nu_t, r_t, t) && (63)\\ p_t &= e^{-\int_0^T r_t dt} P(r_t, t) \end{aligned}$$

and (62) holds for all real world processes S_t, ν_t and r_t which are given by (52) (and which are to be substituted on the right hand side of (63)). In particular, the option price

$$\text{option price} = V(S,\nu,r,0) \qquad (64)$$

given by the solution of (58,69), is independent of μ, m and α but only depends on $\tilde{m}, \tilde{\alpha}$ and the vol and correlation parameters.

A Realistic Model with Stoch Vol and Stochastic Rates:

c) Let

$$H = H\big(\{S_t, r_t\}_{0\le t\le T}\big) \tag{65}$$

be the payoff of some exotic option. Then the price of this option is given by

$$\text{price}(H) \;=\; \tilde{\mathsf{E}}\Big[e^{-\int_0^T \tilde{r}_t dt}\, H\big(\{\tilde{S}_t, \tilde{r}_t\}_{0\le t\le T}\big)\Big] \tag{66}$$

where $(\tilde{S}_t, \tilde{\nu}_t, \tilde{r}_t)$ are given by the risk neutral SDE system

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= \tilde{r}_t\, dt + \sqrt{\tilde{\nu}_t}\, d\tilde{B}_t^S \\ d\tilde{\nu}_t &= \tilde{\alpha}(\tilde{\nu}_t, t)\, dt + \beta\sqrt{\tilde{\nu}_t}\, d\tilde{B}_t^\nu \\ d\tilde{r}_t &= \tilde{m}(\tilde{r}_t, t)\, dt + \sigma\, d\tilde{B}_t^r \end{aligned} \tag{67}$$

with correlated Brownian motions

$$\begin{aligned} d\tilde{B}^S \cdot d\tilde{B}^\nu &= \rho_{S,\nu}\, dt \\ d\tilde{B}^S \cdot d\tilde{B}^r &= \rho_{S,r}\, dt \\ d\tilde{B}^\nu \cdot d\tilde{B}^r &= \rho_{\nu,r}\, dt \end{aligned} \tag{68}$$

Here $\tilde{m}$ and $\tilde{\alpha}$ are the universal functions (55,56).