Chapter 7

The Black-Scholes Equation

In chapter 5, we approximated the Black-Scholes model

$$dS_t/S_t \;=\; \mu\, dt + \sigma\, dx_t \tag{7.1}$$

with a suitable Binomial model and were able to derive a pricing formula for option payoffs $H = H(S_T)$. The time 0 theoretical fair value is given by

$$V_0^{\rm BS} \;=\; e^{-rT} \int_{\mathbb{R}} H\bigl(S_0 e^{\sigma\sqrt{T}y} e^{(r-\frac{\sigma^2}{2})T}\bigr)\, \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2} dy \tag{7.2}$$

One can show that this quantity can also be obtained as the unique solution of the following partial differential equation: $V_0^{\rm BS} = V(S = S_0, t = 0)$ where $V(S,t)$ is a solution of

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \;=\; 0 \tag{7.3}$$

$$V(S, t = T) \;=\; H(S) \tag{7.4}$$

Equation (7.3) is called the Black-Scholes equation. Instead of doing just a brute force calculation and checking that indeed (7.2) is a solution of (7.3), which would give no further insight in the origin of (7.3), we will now derive (7.3,7.4) as the contiuous time limit of the recursion relations for the replicating portfolio values in the approximating Binomial model. Recall from Theorem 2.1 that the replicating strategy δ_{t_k} and the portfolio values V_{t_k} can be inductively (from $k = N$ to $k = 0$) calculated through the following formulae:

$$V_{t_k} \;=\; \frac{(1 - d_{\Delta t} - d_{\Delta t}\mathrm{ret}_{\rm down})V_{t_{k+1}}^{\rm up} \;-\; (1 - d_{\Delta t} - d_{\Delta t}\mathrm{ret}_{\rm up})V_{t_{k+1}}^{\rm down}}{\mathrm{ret}_{\rm up} - \mathrm{ret}_{\rm down}} \tag{7.5}$$

$$V_{t_N} \;=\; H$$

where

$$V_{t_{k+1}}^{\rm up,down} \;:=\; V_{t_{k+1}}\bigl(S_{t_k}(1 + \mathrm{ret}_{\rm up,down})\bigr) \tag{7.6}$$

and, with time steps $t_k = k\Delta t$, the discount factor $d_{k,k+1} = e^{-r(t_{k+1}-t_k)}$ of Theorem 2.1 becomes

$$d_{k,k+1} \;=\; e^{-r\Delta t} \tag{7.7}$$

The delta's are obtained from

$$\delta_{t_k} \;=\; \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mathrm{ret_{up}})\big) \;-\; V_{t_{k+1}}\big(S_{t_k}(1+\mathrm{ret_{down}})\big)}{S_{t_k}(1+\mathrm{ret_{up}}) - S_{t_k}(1+\mathrm{ret_{down}})} \tag{7.8}$$

The Binomial model which approximates the Black-Scholes model (7.1) is given by

$$\mathrm{ret_{up}} \;=\; \mu\Delta t + \sigma\sqrt{\Delta t} \tag{7.9}$$

$$\mathrm{ret_{down}} \;=\; \mu\Delta t - \sigma\sqrt{\Delta t} \tag{7.10}$$

The delta's of (7.8) simply become

$$\delta_t \;=\; \frac{\partial V(S_t,t)}{\partial S_t} \tag{7.11}$$

in the contiuum limit $\Delta t \to 0$. Now let us consider the continuum limit of (7.5). To get a feeling for the problem, let us first put the interest rates to zero, $r = 0$. In that case (7.5) reduces to

$$\begin{aligned} V_{t_k} \;&=\; \frac{-\mathrm{ret_{down}} V_{t_{k+1}}^{\mathrm{up}} + \mathrm{ret_{up}} V_{t_{k+1}}^{\mathrm{down}}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}} \\ &=\; \frac{(-\mu\Delta t + \sigma\sqrt{\Delta T})V_{t_{k+1}}^{\mathrm{up}} + (\mu\Delta t + \sigma\sqrt{\Delta T})V_{t_{k+1}}^{\mathrm{down}}}{2\sigma\sqrt{\Delta t}} \\ &=\; \frac{V_{t_{k+1}}^{\mathrm{up}} + V_{t_{k+1}}^{\mathrm{down}}}{2} \;-\; \mu\Delta t\, \frac{V_{t_{k+1}}^{\mathrm{up}} - V_{t_{k+1}}^{\mathrm{down}}}{2\sigma\sqrt{\Delta t}} \end{aligned} \tag{7.12}$$

Motivated by the Black-Scholes equation where a term $\frac{\partial V}{\partial t}$ shows up, we subtract on both sides of (7.12) the term $V_{t_{k+1}}(S_{t_k})$,

$$\begin{aligned} V_{t_k}(S_{t_k}) - V_{t_{k+1}}(S_{t_k}) \;&=\; \frac{V_{t_{k+1}}^{\mathrm{up}} + V_{t_{k+1}}^{\mathrm{down}}}{2} \;-\; \mu\Delta t\, \frac{V_{t_{k+1}}^{\mathrm{up}} - V_{t_{k+1}}^{\mathrm{down}}}{2\sigma\sqrt{\Delta t}} \;-\; V_{t_{k+1}}(S_{t_k}) \\ &=\; \frac{V_{t_{k+1}}^{\mathrm{up}} - 2V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t)\big) + V_{t_{k+1}}^{\mathrm{down}}}{2} \;-\; \mu\Delta t\, \frac{V_{t_{k+1}}^{\mathrm{up}} - V_{t_{k+1}}^{\mathrm{down}}}{2\sigma\sqrt{\Delta t}} \\ &\quad + V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t)\big) - V_{t_{k+1}}(S_{t_k}) \end{aligned} \tag{7.13}$$

We devide this by Δt and obtain

$$\frac{V_{t_k}(S_{t_k}) - V_{t_{k+1}}(S_{t_k})}{\Delta t} \;=\; \mathrm{term}_1 \;+\; \mathrm{term}_2 \;+\; \mathrm{term}_3 \tag{7.14}$$

with the following quantities:

$$\begin{aligned}
\text{term}_1 &= \frac{V^{\text{up}}_{t_{k+1}} - 2V_{t_{k+1}}\left(S_{t_k}(1+\mu\Delta t)\right) + V^{\text{down}}_{t_{k+1}}}{2\Delta t} \\
&= \frac{\sigma^2 S_{t_k}^2}{2} \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t+\sigma\sqrt{\Delta t})\big) - 2V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t)\big) + V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t-\sigma\sqrt{\Delta t})\big)}{(S_{t_k}\sigma\sqrt{\Delta t})^2} \\
&\overset{\Delta t\to 0}{\to} \frac{\sigma^2 S_{t_k}^2}{2} \frac{\partial^2 V}{(\partial S_{t_k})^2}
\end{aligned} \tag{7.15}$$

$$\begin{aligned}
\text{term}_2 &= -\mu \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t+\sigma\sqrt{\Delta t})\big) - V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t-\sigma\sqrt{\Delta t})\big)}{2\sigma\sqrt{\Delta t}} \\
&= -\mu S_{t_k} \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t+\sigma\sqrt{\Delta t})\big) - V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t-\sigma\sqrt{\Delta t})\big)}{2S_{t_k}\sigma\sqrt{\Delta t}} \\
&\overset{\Delta t\to 0}{\to} -\mu S_{t_k} \frac{\partial V}{\partial S_{t_k}}
\end{aligned} \tag{7.16}$$

$$\begin{aligned}
\text{term}_3 &= \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t)\big) - V_{t_{k+1}}(S_{t_k})}{\Delta t} \\
&= S_{t_k}\mu \frac{V_{t_{k+1}}\big(S_{t_k}(1+\mu\Delta t)\big) - V_{t_{k+1}}(S_{t_k})}{S_{t_k}\mu\Delta t} \\
&\overset{\Delta t\to 0}{\to} \mu S_{t_k} \frac{\partial V}{\partial S_{t_k}}
\end{aligned} \tag{7.17}$$

Thus, with the notation $V = V(S_t, t)$ instead of $V_{t_k}(S_{t_k})$, we get

$$\begin{aligned}
-\frac{\partial V(S_t,t)}{\partial t} &= \lim_{\Delta t\to 0} \frac{V_{t_k}(S_{t_k}) - V_{t_{k+1}}(S_{t_k})}{\Delta t} \\
&= \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 V(S_t,t)}{\partial S_t^2} - \mu S_t \frac{\partial V(S_t,t)}{\partial S_t} + \mu S_t \frac{\partial V(S_t,t)}{\partial S_t}
\end{aligned}$$

or

$$\frac{\partial V(S_t,t)}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 V(S_t,t)}{\partial S_t^2} = 0 \tag{7.18}$$

which is the Black-Scholes equation for zero interest rates. To obtain the Black-Scholes equation with nonzero interest rates, we rewrite (7.5) as follows:

$$\begin{aligned}
V_{t_k} &= \frac{(1 - d_{\Delta t} - d_{\Delta t}\text{ret}_{\text{down}})V^{\text{up}}_{t_{k+1}} - (1 - d_{\Delta t} - d_{\Delta t}\text{ret}_{\text{up}})V^{\text{down}}_{t_{k+1}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\
&= \frac{-\text{ret}_{\text{down}}V^{\text{up}}_{t_{k+1}} + \text{ret}_{\text{up}}V^{\text{down}}_{t_{k+1}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\
&\quad + \frac{\left(1 - d_{\Delta t} - (d_{\Delta t}-1)\text{ret}_{\text{down}}\right)V^{\text{up}}_{t_{k+1}} - \left(1 - d_{\Delta t} - (d_{\Delta t}-1)\text{ret}_{\text{up}}\right)V^{\text{down}}_{t_{k+1}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}
\end{aligned} \tag{7.19}$$

The first term in (7.19) is the contribution for zero interest rates and has been considered following (7.12). The second term in (7.19),

$$(1 - d_{\Delta t}) \times \frac{(1 + \mathrm{ret}_{\mathrm{down}}) V^{\mathrm{up}}_{t_{k+1}} \; - \; (1 + \mathrm{ret}_{\mathrm{up}}) V^{\mathrm{down}}_{t_{k+1}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \tag{7.20}$$

is new. Thus, for non zero interest rates (7.14) changes to

$$\frac{V_{t_k}(S_{t_k}) - V_{t_{k+1}}(S_{t_k})}{\Delta t} \; = \; \mathrm{term}_1 \; + \; \mathrm{term}_2 \; + \; \mathrm{term}_3 \; + \; \mathrm{term}_4 \tag{7.21}$$

with a fourth term given by

$$\begin{aligned}
\mathrm{term}_4 \; &= \; \frac{1 - d_{\Delta t}}{\Delta t} \times \left\{ \frac{V^{\mathrm{up}}_{t_{k+1}} \; - \; V^{\mathrm{down}}_{t_{k+1}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \; + \; \frac{\mathrm{ret}_{\mathrm{down}} \, V^{\mathrm{up}}_{t_{k+1}} \; - \; \mathrm{ret}_{\mathrm{up}} \, V^{\mathrm{down}}_{t_{k+1}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \right\} \\
&= \; \frac{1 - e^{-r\Delta t}}{\Delta t} \times \left\{ \frac{V_{t_{k+1}}\big(S_{t_k}(1 + \mu\Delta t + \sigma\sqrt{\Delta t})\big) \; - \; V_{t_{k+1}}\big(S_{t_k}(1 + \mu\Delta t - \sigma\sqrt{\Delta t})\big)}{2\sigma\sqrt{\Delta t}} \right. \\
&\qquad \left. + \; \frac{(\mu\Delta t - \sigma\sqrt{\Delta t}) \, V^{\mathrm{up}}_{t_{k+1}} \; - \; (\mu\Delta t + \sigma\sqrt{\Delta t}) \, V^{\mathrm{down}}_{t_{k+1}}}{2\sigma\sqrt{\Delta t}} \right\} \\
&= \; \frac{1 - e^{-r\Delta t}}{\Delta t} \times \left\{ S_{t_k} \, \frac{V_{t_{k+1}}\big(S_{t_k}(1 + \mu\Delta t + \sigma\sqrt{\Delta t})\big) \; - \; V_{t_{k+1}}\big(S_{t_k}(1 + \mu\Delta t - \sigma\sqrt{\Delta t})\big)}{2 S_{t_k} \sigma\sqrt{\Delta t}} \right. \\
&\qquad \left. + \; \mu\Delta t \frac{V^{\mathrm{up}}_{t_{k+1}} \; - \; V^{\mathrm{down}}_{t_{k+1}}}{2\sigma\sqrt{\Delta t}} \; - \; \frac{V^{\mathrm{up}}_{t_{k+1}} \; + \; V^{\mathrm{down}}_{t_{k+1}}}{2} \right\} \\
&\overset{\Delta t \to 0}{\to} \; r \times \left\{ S_{t_k} \, \frac{\partial V}{\partial S_{t_k}} \; + \; 0 \; - \; V \right\}
\end{aligned} \tag{7.22}$$

and (7.21) becomes

$$-\frac{\partial V(S_t, t)}{\partial t} \; = \; \frac{\sigma^2 S_t^2}{2} \, \frac{\partial^2 V(S_t, t)}{\partial S_t^2} \; + \; r \, S_t \, \frac{\partial V(S_t, t)}{\partial S_t} \; - \; r \, V \tag{7.23}$$

which is the Black-Scholes equation (7.3) with non zero interest rates.

Reduction of the Black-Scholes Equation to the Diffusion Equation

Suppose that we would not know that the solution to the Black-Scholes equation is given by (7.2), how would we proceed from (7.3,7.4) to obtain a solution? One possibility is to transform the Black-Scholes equation into the diffusion equation $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$ which can be solved, for example with Fourier transform, in a pretty straightforward way. This calculation goes as follows:

The Black-Scholes equation for a european option with payoff $H(S_T)$ reads

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{7.24}$$

with the final condition

$$V(S,T) = H(S) \tag{7.25}$$

To turn (7.24) into a constant coefficient equation, we start by introducing the variables x and τ according to

$$S = e^{\sigma x}, \qquad \tau = T - t \tag{7.26}$$

and write

$$V(S,t) = V(e^{\sigma x}, T-\tau) = e^{-r\tau} v(x,\tau) \tag{7.27}$$

Because of $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau}$ and

$$\frac{\partial}{\partial S} = \frac{1}{\sigma S}\frac{\partial}{\partial x}, \qquad \frac{\partial^2}{\partial S^2} = -\frac{1}{\sigma S^2}\frac{\partial}{\partial x} + \frac{1}{\sigma^2 S^2}\frac{\partial^2}{\partial x^2} \tag{7.28}$$

(7.24) becomes

$$\begin{aligned} rv - \frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2} S^2 \left(-\frac{1}{\sigma S^2}\frac{\partial v}{\partial x} + \frac{1}{\sigma^2 S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\frac{1}{\sigma S}\frac{\partial v}{\partial x} - rv &= 0 \\ \Leftrightarrow \qquad \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial \tau} \end{aligned} \tag{7.29}$$

This looks almost like the diffusion equation. To eliminate the first derivative with respect to x we put

$$k := \frac{r}{\sigma} - \frac{\sigma}{2} \tag{7.30}$$

and make the ansatz

$$v(x,\tau) = e^{-\alpha x - \beta\tau} u(x,\tau) \tag{7.31}$$

which gives

$$\begin{aligned} \frac{1}{2}\Big(\alpha^2 u - 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\Big) + k\Big(-\alpha u + \frac{\partial u}{\partial x}\Big) &= -\beta u + \frac{\partial u}{\partial \tau} \\ \Leftrightarrow \qquad \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (-\alpha + k)\frac{\partial u}{\partial x} + \Big(\frac{\alpha^2}{2} - \alpha k + \beta\Big)u &= \frac{\partial u}{\partial \tau} \end{aligned} \tag{7.32}$$

With the choice

$$\alpha = k = \frac{r}{\sigma} - \frac{\sigma}{2}, \qquad \beta = \frac{k^2}{2} = \frac{1}{2}\Big(\frac{r^2}{\sigma^2} - r + \frac{\sigma^2}{4}\Big) \tag{7.33}$$

we have to solve the diffusion equation

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} \tag{7.34}$$

with the initial condition

$$u(x,0) = e^{kx} H(e^{\sigma x}) \tag{7.35}$$

The solution is

$$u(x,\tau) = \int_{\mathbb{R}} e^{ky} H(e^{\sigma y}) \, \tfrac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-y)^2}{2\tau}} \, dy \tag{7.36}$$

and we arrive at ($S = e^{\sigma x}$, $t = T - \tau$)

$$\begin{aligned}
V(S,t) &= e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2\tau} \int_{\mathbb{R}} e^{(\frac{r}{\sigma}-\frac{\sigma}{2})(y-x)} \, H(e^{\sigma y}) \, \tfrac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-y)^2}{2\tau}} \, dy \\
&= \tfrac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2\tau} \int_{\mathbb{R}} H(Se^{\sigma\sqrt{\tau}y}) \, e^{-\frac{y^2}{2}+(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\tau}y} dy \\
&= \tfrac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2\tau} \int_{\mathbb{R}} H(Se^{\sigma\sqrt{\tau}y}) e^{-\frac{1}{2}(y^2-2(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\tau}y+(\frac{r}{\sigma}-\frac{\sigma}{2})^2\tau)} dy \;\; e^{\frac{1}{2}(\frac{r}{\sigma}-\frac{\sigma}{2})^2\tau} \\
&= \tfrac{1}{\sqrt{2\pi}} \, e^{-r\tau} \int_{\mathbb{R}} H(Se^{\sigma\sqrt{\tau}y}) e^{-\frac{1}{2}(y-(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\tau})^2} dy \\
&= \tfrac{1}{\sqrt{2\pi}} \, e^{-r\tau} \int_{\mathbb{R}} H\big(S \, e^{\sigma\sqrt{\tau}\, y + (r-\frac{\sigma^2}{2})\tau}\big) \, e^{-\frac{y^2}{2}} \, dy
\end{aligned} \tag{7.37}$$

which coincides with (7.2).