Chapter 2

The Binomial Model

Let $S$ be some tradable asset with prices

$$S_k = S(t_k), \quad k = 0, 1, 2, \ldots \quad (2.1)$$

and let

$$H = H(S_0, S_1, ..., S_{N-1}, S_N) \quad (2.2)$$

be some option payoff with start date $t_0$ and end date or maturity $t_N$. We want to replicate the option payoff (2.2) with a suitable trading strategy in the underlying $S$. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding $\delta_k$ assets at the end of day $t_k$ generates the amount

$$V_N = V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \quad (2.3)$$

Each $\delta_k$ will be determined on the end of trading day $t_k$. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, $\delta_k$ can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k = \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \quad (2.4)$$

**Definition 2.1:** We say that an option payoff (2.2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of $\delta_k$ of the form (2.4) and some initial amount $V_0$ such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) = V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \quad (2.5)$$

The initial amount $V_0$ which is needed to set up the replicating strategy is called the theoretical fair value of $H$.  

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Now let us consider the question to what extent replication of options is possible. Equation (2.5) can be rewritten as

$$H(S_0, S_1, \ldots, S_{N-1}, S_N) = V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$H(S_0, S_1, \ldots, S_{N-1}, S_N) - \delta_{N-1}S_N = V_{N-1} - \delta_{N-1}S_{N-1} = \text{some function of } S_0, S_1, \ldots, S_{N-1} \tag{2.6}$$

That is, the right hand side of (2.6) is independent of $S_N$. Let us introduce the return of the asset $S$ from $t_{k-1}$ to $t_k$,

$$\text{ret}_k := \frac{S_k - S_{k-1}}{S_{k-1}} = \frac{S_k}{S_{k-1}} - 1 \tag{2.7}$$

such that

$$S_k = S_{k-1}(1 + \text{ret}_k) \tag{2.8}$$

Then equation (2.6) can be rewritten as

$$H(S_0, S_1, \ldots, S_{N-1}, S_{N-1}(1 + \text{ret}_N)) - \delta_{N-1}S_{N-1}(1 + \text{ret}_N) = \text{const} \tag{2.9}$$

where the const in the equation above means that the left hand side of (2.9) has to be the same for all possible choices of $\text{ret}_N$. Since there is only 1 free parameter in (2.9), namely $\delta_{N-1}$, we can only allow for 2 possible choices for $\text{ret}_N$, say,

$$\text{ret}_N \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \tag{2.10}$$

and in that case we have to have

$$H(S_0, S_1, \ldots, S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})) - \delta_{N-1}S_{N-1}(1 + \text{ret}_{\text{up}}) =$$

$$H(S_0, S_1, \ldots, S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}})) - \delta_{N-1}S_{N-1}(1 + \text{ret}_{\text{down}})$$

which determines $\delta_{N-1}$ to

$$\delta_{N-1} = \frac{H(S_0, \ldots, S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})) - H(S_0, \ldots, S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}}))}{S_{N-1}(1 + \text{ret}_{\text{up}}) - S_{N-1}(1 + \text{ret}_{\text{down}})} \tag{2.11}$$

Thus, if we allow for 2 possible choices of returns only as in (2.10), replication of option payoffs seem to be possible. This leads to the following
**Definition 2.2:** If the price process $S_k = S(t_k)$ of some tradable asset $S$ has the dynamics

$$S_k = S_{k-1}(1 + \text{ret}_k) \quad \text{with} \quad \text{ret}_k \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \quad \forall \ k = 1, 2, ...$$  \hspace{1cm} (2.12)

then we say that $S$ is given by the Binomial model.

**Remark:** Observe that in Definition 2.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability $p_{\text{up}}$ such that an up-return $\text{ret}_{\text{up}}$ will occur and a probability $p_{\text{down}} = 1 - p_{\text{up}}$ for the occurrence of a down-return. We did that because the replicating strategy and the theoretical option fair value $V_0$ are actually independent of such probabilities. Nevertheless we have to remark that the definition of the Binomial model as given in the standard literature usually includes a specification of $p_{\text{up}}$ and $p_{\text{down}} = 1 - p_{\text{up}}$.

Now we are in a position to formulate the following important theorem:

**Theorem 2.1:** Let $S$ be some tradable asset whose price process is given by the Binomial model (2.12). Let $r \geq 0$ denote some constant interest rate. Then every option payoff $H = H(S_0, ..., S_N)$ can be replicated. A replicating strategy is given by, for $k = 0, 1, ... , N - 1$:

$$\delta_k = \frac{V_{k+1}(S_0, ..., S_k, S_k(1 + \text{ret}_{\text{up}})) - V_{k+1}(S_0, ..., S_k, S_k(1 + \text{ret}_{\text{down}}))}{S_k(1 + \text{ret}_{\text{up}}) - S_k(1 + \text{ret}_{\text{down}})}$$  \hspace{1cm} (2.13)

and the portfolio values $V_k$, including the theoretical fair value $V_0$, can be inductively calculated through the formulae

$$V_k = \frac{(1 - d_{k,k+1} - d_{k,k+1}\text{ret}_{\text{down}})V_{k+1}^{\text{up}} - (1 - d_{k,k+1} - d_{k,k+1}\text{ret}_{\text{up}})V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}$$  \hspace{1cm} (2.14)

$$V_N = H$$

where we used the abbreviations

$$V_{k+1}^{\text{up,down}} := V_{k+1}(S_0, ..., S_k, S_k(1 + \text{ret}_{\text{up,down}}))$$  \hspace{1cm} (2.15)

and introduced the discount factor

$$d_{k,k+1} := e^{-r(t_{k+1}-t_k)}$$  \hspace{1cm} (2.16)

**Proof:** For nonzero interest rates we have

$$v_N = v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$
which is equivalent to
\[ v_{k+1} = v_k + \delta_k (s_{k+1} - s_k) \quad \forall k = 0, 1, ..., N - 1 \]
or, since \( v_k = e^{-r(t_k-t_0)}V_k \),
\[ e^{-r(t_{k+1}-t_k)}V_{k+1} = V_k + \delta_k \left( e^{-r(t_{k+1}-t_k)}S_k(1 + \text{ret}_k) - S_k \right) \]

Since we assume the price dynamics of the Binomial model, this equation is equivalent to
\[
\begin{align*}
e^{-r(t_{k+1}-t_k)}V^\text{up}_{k+1} &= V_k + \delta_k \left( e^{-r(t_{k+1}-t_k)}S_k(1 + \text{ret}_{\text{up}}) - S_k \right) \quad (2.17) \\
e^{-r(t_{k+1}-t_k)}V^\text{down}_{k+1} &= V_k + \delta_k \left( e^{-r(t_{k+1}-t_k)}S_k(1 + \text{ret}_{\text{down}}) - S_k \right) \quad (2.18)
\end{align*}
\]

Subtracting (2.18) from (2.17) gives
\[ e^{-r(t_{k+1}-t_k)}(V^\text{up}_{k+1} - V^\text{down}_{k+1}) = \delta_k \left( e^{-r(t_{k+1}-t_k)}S_k(1 + \text{ret}_{\text{up}}) - e^{-r(t_{k+1}-t_k)}S_k(1 + \text{ret}_{\text{down}}) \right) \]
or
\[ \delta_k = \frac{V^\text{up}_{k+1} - V^\text{down}_{k+1}}{S_k(1 + \text{ret}_{\text{up}}) - S_k(1 + \text{ret}_{\text{down}})} \quad (2.19) \]

which coincides with (2.13). Substituting this value of \( \delta_k \) in equation (2.17) and solving for \( V_k \) gives (now using the abbreviation (2.16) for the discount factors)
\[ d_{k,k+1}V^\text{up}_{k+1} - \delta_k S_k(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1) = V_k \quad \Leftrightarrow \]

\[
\begin{align*}
V_k &= d_{k,k+1}V^\text{up}_{k+1} - \frac{V^\text{up}_{k+1} - V^\text{down}_{k+1}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \left( d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \right) \\
&= d_{k,k+1}V^\text{up}_{k+1}(\text{ret}_{\text{up}} - \text{ret}_{\text{down}}) - V^\text{up}_{k+1}(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1) + V^\text{down}_{k+1}(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1) \\
&= -V^\text{up}_{k+1}(d_{k,k+1}(1 + \text{ret}_{\text{down}}) - 1) + V^\text{down}_{k+1}(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1) \\
&= \frac{V^\text{up}_{k+1}(1 - d_{k,k+1}(1 + \text{ret}_{\text{down}})) - V^\text{down}_{k+1}(1 - d_{k,k+1}(1 + \text{ret}_{\text{up}}))}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\
\end{align*}
\]

and this coincides with (2.14). ■

**Remarks:** 1) If \( H \) is some (then usually called “european”) option which depends only on the underlying price at maturity,
\[
H = H(S_N) \quad (2.20)
\]
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then the $\delta_k$ and the value of the replicating portfolio $V_k$ at $t_k$ depend only on the asset price $S_k$ and do not depend on earlier prices $S_{k-1}, S_{k-2}, ..., S_0$. That is,

\begin{align*}
V_k &= V_k(S_k) \\
\delta_k &= \delta_k(S_k)
\end{align*}

(2.21) (2.22)

2) Assume zero interest rates such that $d_{k,k+1} = 1$. Then (2.14) becomes

\begin{align*}
V_k &= -\text{ret}_{\text{down}} V_{k+1}^{\text{up}} + \text{ret}_{\text{up}} V_{k+1}^{\text{down}} \\
&= \frac{V_{k+1}^{\text{up}} + V_{k+1}^{\text{down}}}{2} - \frac{\text{ret}_{\text{up}} + \text{ret}_{\text{down}}}{2} \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}
\end{align*}

(2.23)

If we further assume a “symmetric” Binomial model with $\text{ret}_{\text{up}} = q\%$ and $\text{ret}_{\text{down}} = -q\%$ we obtain the simple recursion formula

\begin{align*}
V_k &= \frac{V_{k+1}^{\text{up}} + V_{k+1}^{\text{down}}}{2}
\end{align*}

(2.24)