Chapter 13

The Time-Dependent Black-Scholes Model and Calibration to Market

The time-dependent Black-Scholes model is given by the following stochastic differential equation (SDE) for an underlying asset price process:

$$\frac{dS_t}{S_t} \;=\; \mu_t\,dt \,+\, \sigma_t\,dx_t \tag{13.1}$$

where μ_t and σ_t are deterministic (that is, non-stochastic) functions of time t. Let us first convince ourselves that exact payoff replication in this model is still possible. For the time-independent Black-Scholes model introduced in Chapter 4 and 5 we did that by approximating the model through a suitable Binomial model and using the fact the arbitrary payoffs can be replicated in the Binomial model. Here we will follow a different path and show this directly in continuous time by using the continuous time formalism introduced in Chapter 8.

Let us start by proving again that exact payoff replication is possible for the time-independent Black-Scholes model given by

$$\frac{dS_t}{S_t} \;=\; \mu\,dt \,+\, \sigma\,dx_t \tag{13.2}$$

The time-dependent case (13.1) will then be a straightforward generalization. Our standard equation of Chapter 1 for the value of a replicating portfolio,

$$v_{t_k} \;=\; v_{t_0} + \sum_{j=1}^{k} \delta_{t_{j-1}}(s_{t_j} - s_{t_{j-1}})$$

reads in continuous time

$$v_t \;=\; v_{t_0} + \int_{t_0}^{t} \delta_u\,ds_u \tag{13.3}$$

where the stochastic integral on the right hand side of (13.3) is an Ito-integral. From the analysis of Chapter 7 we know that the number of stocks δ_t to be hold at time t must be equal to

$$\delta_t \quad = \quad \frac{\partial V(S_t, t)}{\partial S_t} \tag{13.4}$$

where the undiscounted value $V = V(S_t, t)$ of the replicating portfolio at time t, or equivalently, the option price at time t, is a solution of the Black-Scholes partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \quad = \quad 0 \tag{13.5}$$

with final condition

$$V(S_T, t = T) \quad = \quad H(S_T) \tag{13.6}$$

Thus the statement "exact payoff replication is possible in the Black-Scholes model" is equivalent (actually at this point only for non path-dependent payoffs) to the following mathematical statement:

Theorem 13.1: Let $V = V(S_t, t)$ be the solution to (13.5,13.6) and let $\delta_t = \delta_t(S_t)$ be given by (13.4). Now let $S_t^{(\mu)}$ be any stochastic path realization of the time-independent Black-Scholes model given by (13.2) and let $s_t^{(\mu)} = e^{-rt} S_t^{(\mu)}$ denote the discounted realized price path (we assume $t_0 = 0$). Then we have for any such stochastic path realization:

$$e^{-rT}\, H(S_T^{(\mu)}) \quad = \quad V_0 \; + \; \int_0^T \delta_t(S_t^{(\mu)})\, ds_t^{(\mu)} \tag{13.7}$$

with $V_0 = V(S_0^{(\mu)}, t = 0)$ being the option price for the payoff H.

Proof: Let $V = V(S_t, t)$ be the solution to (13.5,13.6). Because of

$$\begin{aligned} V(S_0, 0) \quad &= \quad V_0 && (13.8) \\ V(S_T, T) \quad &= \quad H(S_T) \\ v(S_T, T) = e^{-rT} V(S_T, T) \quad &= \quad e^{-rT} H(S_T) =: h(S_T) && (13.9) \end{aligned}$$

we can write

$$\begin{aligned} e^{-rT} H(S_T) - V_0 \quad &= \quad e^{-rT} V(S_T, T) - e^{-r0} V(S_0, 0) \\ &= \quad v(S_T, T) - v(S_0, 0) \\ &= \quad \textstyle\int_0^T dv(S_t, t) && (13.10) \end{aligned}$$

with

$$dv(S_t,t) \;=\; v(S_{t+dt},t+dt) - v(S_t,t) \tag{13.11}$$

The function $v = e^{-rt}V$ is given by a product of a deterministic function e^{-rt} and a stochastic function V (because the S which is put into V is stochastic). In that case, the usual product rule of differentiation from ordinary calculus can be used (or more precisely, the covariation $d(e^{-rt}) \cdot dV = 0$ vanishes) to obtain

$$\begin{aligned} dv &= d(e^{-rt}V) \\ &= d(e^{-rt})\,V + e^{-rt}\,dV \\ &= -r\,e^{-rt}\,dt\,V + e^{-rt}\,dV \\ &= -r\,v\,dt + e^{-rt}\,dV \end{aligned} \tag{13.12}$$

The quantity

$$dV \;=\; dV(S_t,t) \;=\; V(S_{t+dt},t+dt) - V(S_t,t)$$

on the right hand side of (13.12) has to be calculated with the Ito-Formula from Chapter 8, Theorem 8.1 (or the slight generalization (8.36), because of the extra t-dependence of V): We have

$$\begin{aligned} dV(S_t,t) &= \tfrac{\partial V}{\partial S_t} dS_t + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + \tfrac{\partial V}{\partial t} dt \\ &\overset{(13.4)}{=} \delta_t\, dS_t + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + \tfrac{\partial V}{\partial t} dt \end{aligned} \tag{13.13}$$

or

$$\begin{aligned} e^{-rt}\,dV &= \delta_t\, e^{-rt}\, dS_t + e^{-rt}\,\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + e^{-rt}\,\tfrac{\partial V}{\partial t} dt \\ &= \delta_t\left[e^{-rt}\,dS_t + d(e^{-rt})\,S_t\right] - \delta_t\, d(e^{-rt})\,S_t + e^{-rt}\,\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + e^{-rt}\,\tfrac{\partial V}{\partial t} dt \\ &= \delta_t\, ds_t - \delta_t\, d(e^{-rt})\,S_t + e^{-rt}\,\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + e^{-rt}\,\tfrac{\partial V}{\partial t} dt \end{aligned}$$

such that from (13.12) we get

$$\begin{aligned} dv &= d(e^{-rt})\,V + e^{-rt}\,dV \\ &= \delta_t\, ds_t - \delta_t\, d(e^{-rt})\,S_t + e^{-rt}\,\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S_t^2}(dS_t)^2 + \left[\tfrac{\partial e^{-rt}}{\partial t}V + e^{-rt}\tfrac{\partial V}{\partial t}\right]dt \\ &= \delta_t\, ds_t - \tfrac{\partial V}{\partial S_t}\,(-r)e^{-rt}\,S_t\,dt + \tfrac{1}{2}\tfrac{\partial^2 v}{\partial S_t^2}(dS_t)^2 + \tfrac{\partial v}{\partial t}\,dt \\ &= \delta_t\, ds_t + \tfrac{1}{2}\tfrac{\partial^2 v}{\partial S_t^2}(dS_t)^2 + \left[\tfrac{\partial v}{\partial t} + rS_t\,\tfrac{\partial v}{\partial S_t}\right]dt \end{aligned} \tag{13.14}$$

Observe that equation (13.14) holds for any stochastic process S_t and any function $v = e^{-rt}V$, so far we have not used the Black-Scholes dynamics (13.2) or the Black-Scholes PDE (13.5).

Now suppose that S_t is given by the Black-Scholes dynamics (13.2). Then we have

$$\left(\tfrac{dS_t}{S_t}\right)^2 \;=\; \sigma^2\,(dx_t)^2 \;=\; \sigma^2\,dt \tag{13.15}$$

and (13.14) becomes

$$dv \;=\; \delta_t\,ds_t \;+\; \left[\tfrac{\sigma^2}{2}S_t^2\tfrac{\partial^2 v}{\partial S_t^2} + \tfrac{\partial v}{\partial t} + rS_t\,\tfrac{\partial v}{\partial S_t}\right]dt \tag{13.16}$$

The square bracket in (13.16) is identical to

$$e^{-rt}\left[\frac{\sigma^2}{2}S_t^2\frac{\partial^2 V}{\partial S_t^2} + \frac{\partial V}{\partial t} - rV + rS_t\,\frac{\partial V}{\partial S_t}\right] \;\stackrel{(13.5)}{=}\; 0 \tag{13.17}$$

and thus we end up with

$$\begin{aligned} e^{-rT}H(S_T) - V_0 \;&\stackrel{(13.10)}{=}\; \textstyle\int_0^T dv(S_t,t) \\ &\stackrel{(13.16)}{=}\; \textstyle\int_0^T \delta_t\,ds_t \;+\; \int_0^T\left[\tfrac{\sigma^2}{2}S_t^2\tfrac{\partial^2 v}{\partial S_t^2} + \tfrac{\partial v}{\partial t} + rS_t\,\tfrac{\partial v}{\partial S_t}\right]dt \\ &\stackrel{(13.17)}{=}\; \textstyle\int_0^T \delta_t\,ds_t \end{aligned} \tag{13.18}$$

which proves the theorem. ■

By going through the above proof, the following statement follows immediately:

Corollary 13.2: Let $V = V(S_t,t)$ be the solution of the time-dependent Black-Scholes PDE given by

$$\frac{\partial V}{\partial t} + \frac{\sigma_t^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \;=\; 0 \tag{13.19}$$

with some time dependent volatility function σ_t, with final condition

$$V(S_T, t=T) \;=\; H(S_T) \tag{13.20}$$

and let $\delta_t = \delta_t(S_t)$ be given by (13.4). Now let $S_t^{(\mu)}$ be any stochastic path realization of the time-dependent Black-Scholes model given by (13.1) and let $s_t^{(\mu)} = e^{-rt}S_t^{(\mu)}$ denote the discounted realized price path (we assume $t_0 = 0$). Then we have for any such stochastic path realization:

$$e^{-rT}\,H(S_T^{(\mu)}) \;=\; V_0 \;+\; \int_0^T \delta_t(S_t^{(\mu)})\,ds_t^{(\mu)} \tag{13.21}$$

with $V_0 = V(S_0^{(\mu)}, t=0)$ being the option price for the payoff H.

In the time-independent Black-Scholes model, the asset price process S_t has an explicit representation:

$$S_t \;=\; e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \tag{13.22}$$

In the time-dependent case, it is still possible to write down an explicit formula, although it looks slightly more complicated:

Lemma 13.3: Let $\{x_t\}_{t\geq 0}$ be a Brownian motion and let $\{S_t\}_{t\geq 0}$ be given by

$$S_t \;=\; e^{\int_0^t \sigma_u \, dx_u + \int_0^t (\mu_u - \frac{\sigma_u^2}{2}) du} \tag{13.23}$$

with some deterministic drift function μ_t and volatility function σ_t. Then S_t is a solution of the SDE (13.1),

$$\frac{dS_t}{S_t} \;=\; \mu_t \, dt \,+\, \sigma_t \, dx_t \; .$$

Proof: Let I_t denote the stochastic integral

$$I_t \;:=\; \int_0^t \sigma_u \, dx_u \tag{13.24}$$

and let F_t abbreviate the deterministic function

$$F_t \;:=\; \int_0^t (\mu_u - \frac{\sigma_u^2}{2}) du \tag{13.25}$$

such that

$$S_t \;=\; e^{I_t + F_t} \tag{13.26}$$

We have

$$dI_t \;=\; \sigma_t \, dx_t \tag{13.27}$$

$$(dI_t)^2 \;=\; (\sigma_t \, dx_t)^2 \;=\; \sigma_t^2 \, dt \tag{13.28}$$

and, since F_t is deterministic,

$$dF_t \;=\; \big(\mu_t - \frac{\sigma_t^2}{2}\big) dt \tag{13.29}$$

$$(dF_t)^2 \;=\; 0$$

$$(dI_t + dF_t)^2 \;=\; (dI_t)^2 \;=\; \sigma_t^2 \, dt \tag{13.30}$$

such that

$$\begin{aligned} dS_t &= d\big(e^{I_t+F_t}\big) \\ &= e^{I_t+F_t}\, d(I_t+F_t) + \tfrac{1}{2}\, e^{I_t+F_t}\, d(I_t+F_t)^2 \\ &= S_t\left[\sigma_t\, dx_t + (\mu_t - \tfrac{\sigma_t^2}{2})dt\right] + \tfrac{1}{2}\, S_t\, \sigma_t^2\, dt \\ &= S_t\left[\sigma_t\, dx_t + \mu_t\, dt\right] \end{aligned} \tag{13.31}$$

which proves the lemma. ■

Our next task is to determine a general pricing formula for options with underlyings which have a time-dependent Black-Scholes dynamics. There is the following analog of Theorem 9.2:

Theorem 13.4: Let $\{x_t\}_{0<t\leq T}$ be a Brownian motion and let

$$S_t^{(\mu)} \;:=\; S_0\, e^{\int_0^t \sigma_u\, dx_u + \int_0^t (\mu_u - \frac{\sigma_u^2}{2})du} \tag{13.32}$$

be an underlying with time-dependent Black-Scholes price dynamics. Let

$$0 \leq t_1 < \cdots < t_m \leq T$$

be some observation times and let $H : \mathbb{R}^m \to \mathbb{R}$ be some option payoff which pays the amount

$$H\big(S_{t_1}^{(\mu)}, \cdots, S_{t_m}^{(\mu)}\big)$$

at maturity T to the option buyer. Then the fair price V_0 at time $t = 0$ of this option is given by the following formula:

$$V_0 \;=\; e^{-rT}\,\mathsf{E}_W\big[H(S_{t_1}^{(r)}, ..., S_{t_m}^{(r)})\big] \tag{13.33}$$

Here $\mathsf{E}_W[\,\cdot\,]$ denotes the expectation value with respect to the standard Wiener measure and $S_t^{(r)}$ is the risk neutral price process given by

$$S_t^{(r)} \;:=\; S_0\, e^{\int_0^t \sigma_u\, dx_u + \int_0^t (r - \frac{\sigma_u^2}{2})du} \tag{13.34}$$

Sketch of Proof: The proof of this theorem is similar to the reasoning of Chapter 9: an equivalent martingale measure $d\tilde{W}$ is constructed such that the discounted price process

$$s_t \;=\; s_t^{(\mu)} \;=\; e^{-rt} S_t^{(\mu)} \;=\; e^{\int_0^t \sigma_u\, dx_u + \int_0^t (\mu_u - r - \frac{\sigma_u^2}{2})du} \tag{13.35}$$

is a martingale with respect to $d\tilde{W}$. That is, the following equation should hold for all $t_1 < t_2$:

$$\mathsf{E}_{\tilde{W}}\left[\, s_{t_2}^{(\mu)} \,|\, x_{t_1} \right] \;=\; s_{t_1}^{(\mu)} \tag{13.36}$$

If this is the case, one has

$$\begin{aligned} \mathsf{E}_{\tilde{W}}\left[\, ds_t^{(\mu)} \,|\, x_t \right] \;&=\; \mathsf{E}_{\tilde{W}}\left[\, s_{t+dt}^{(\mu)} - s_t^{(\mu)} \,|\, x_t \right] \\ &=\; \mathsf{E}_{\tilde{W}}\left[\, s_{t+dt}^{(\mu)} \,|\, x_t \right] \;-\; s_t^{(\mu)} \\ &\overset{(13.36)}{=}\; s_t^{(\mu)} \;-\; s_t^{(\mu)} \;=0 \end{aligned} \tag{13.37}$$

and furthermore

$$\begin{aligned} \mathsf{E}_{\tilde{W}}\left[\delta_t(S_t^{(\mu)})\, ds_t^{(\mu)}\right] \;&=\; \mathsf{E}_{\tilde{W}}\left[\delta_t(S_t^{(\mu)})\, ds_t^{(\mu)} \,|\, x_0 \right] \\ &=\; \mathsf{E}_{\tilde{W}}\left[\mathsf{E}_{\tilde{W}}\left[\delta_t(S_t^{(\mu)})\, ds_t^{(\mu)} \,|\, x_t \right] \,\Big|\, x_0 \right] \\ &=\; \mathsf{E}_{\tilde{W}}\left[\, \delta_t(S_t^{(\mu)}) \underbrace{\mathsf{E}_{\tilde{W}}\left[\, ds_t^{(\mu)} \,|\, x_t \right]}_{=0 \ (13.37)} \,\Big|\, x_0 \right] \\ &=\; 0 \end{aligned} \tag{13.38}$$

such that, if we take the expectation value of the equation (13.21) of Corollary 13.2:

$$\begin{aligned} \mathsf{E}_{\tilde{W}}\left[\, e^{-rT}\, H(S_T^{(\mu)}) \right] \;&=\; \mathsf{E}_{\tilde{W}}\left[\, V_0 \;+\; \int_0^T \delta_t(S_t^{(\mu)})\, ds_t^{(\mu)} \right] \\ \Leftrightarrow \quad e^{-rT}\mathsf{E}_{\tilde{W}}\left[\, H(S_T^{(\mu)}) \right] \;&=\; V_0 \;+\; \int_0^T \underbrace{\mathsf{E}_{\tilde{W}}\left[\delta_t(S_t^{(\mu)})\, ds_t^{(\mu)}\right]}_{=0 \ \ (13.38)} \\ \Leftrightarrow \quad V_0 \;&=\; e^{-rT}\mathsf{E}_{\tilde{W}}\left[\, H(S_T^{(\mu)}) \right] \end{aligned} \tag{13.39}$$

A similar calculation as those in the proof of Theorem 9.2 then shows that

$$\mathsf{E}_{\tilde{W}}\left[\, H(S_T^{(\mu)}) \right] \;=\; \mathsf{E}_{W}\left[\, H(S_T^{(r)}) \right] \tag{13.40}$$

with $S_t^{(r)}$ being the risk neutral price process given by (13.34). Since the actual construction of $d\tilde{W}$ needs the material of Chapter 16 and because we do not need $d\tilde{W}$ in the following, we postpone the construction of $d\tilde{W}$ to a later time. ∎

Let us now specialize our general pricing formula (13.33) to the case of a non path-dependent european option with payoff $H = H(S_T)$. Then we obtain the following theorem, which is the analog of the pricing formula (5.25) of Theorem 5.2 for the time-independent Black-Scholes model:

Theorem 13.5: a) Let $\{x_t\}_{t\geq 0}$ be a Brownian motion, let$W$ be the Wiener measure and let σ_t be some deterministic function. Let F be some function. Then

$$\begin{aligned}\mathsf{E}_W\Big[F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\Big] &= \int F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,dW(\{x_t\}_{0<t\leq T}) \\ &= \int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx \qquad (13.41)\end{aligned}$$

where the implied volatility $\sigma_{\mathrm{imp},T}$ is given by

$$T\,\sigma_{\mathrm{imp},T}^2 := \textstyle\int_0^T \sigma_t^2\,dt \qquad (13.42)$$

b) Let $H = H(S_T)$ be the payoff of some non path-dependent option. Then its fair value $V_0 = V_0^{\mathrm{BSTD}}$ in the time-dependent Black-Scholes model is given by

$$V_0^{\mathrm{BSTD}} = e^{-rT}\int_{\mathbb{R}} H\Big(S_0\,e^{\sigma_{\mathrm{imp},T}\sqrt{T}\,x + (r-\frac{\sigma_{\mathrm{imp},T}^2}{2})T}\Big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx \qquad (13.43)$$

with an implied volatility $\sigma_{\mathrm{imp},T}$ is given by (13.42).

Proof: a) We have

$$\begin{aligned}\mathsf{E}_W\Big[F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\Big] &= \int F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,dW(\{x_t\}_{0<t\leq T}) \\ &= \lim_{\Delta t\to 0}\int F\Big(\sqrt{\Delta t}\sum_{j=1}^{N_T}\sigma_{t_j}\,\phi_j\Big)\prod_{j=1}^{N_T} e^{-\frac{\phi_j^2}{2}}\,\frac{d\phi_j}{\sqrt{2\pi}} \\ &= \lim_{\Delta t\to 0}\int F\Big(\sqrt{\Delta t}\sum_{j=1}^{N_T}\phi_j\Big)\prod_{j=1}^{N_T} e^{-\frac{\phi_j^2}{2\sigma_{t_j}^2}}\,\frac{d\phi_j}{\sqrt{2\pi\sigma_{t_j}^2}} \\ &\stackrel{(4.11)}{=} \lim_{\Delta t\to 0}\int F(y_T)\prod_{j=1}^{N_T} e^{-\frac{(y_{t_j}-y_{t_{j-1}})^2}{2\Delta t\sigma_{t_j}^2}}\,\frac{dy_{t_j}}{\sqrt{2\pi\Delta t\sigma_{t_j}^2}} \qquad (13.44)\end{aligned}$$

Because of (4.15)

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\Delta t\sigma_{t_j}^2}}\,e^{-\frac{(y_{t_j}-y_{t_{j-1}})^2}{2\Delta t\sigma_{t_j}^2}}\;\frac{1}{\sqrt{2\pi\Delta t\sigma_{t_{j+1}}^2}}\,e^{-\frac{(y_{t_{j+1}}-y_{t_j})^2}{2\Delta t\sigma_{t_{j+1}}^2}}\;dy_{t_j} \\ &\quad= \int_{\mathbb{R}} p_{\Delta t\sigma_{t_j}^2}(y_{t_{j-1}},y_{t_j})\,p_{\Delta t\sigma_{t_{j+1}}^2}(y_{t_j},y_{t_{j+1}})\,dy_{t_j} \\ &\quad= p_{\Delta t\sigma_{t_j}^2+\Delta t\sigma_{t_{j+1}}^2}(y_{t_{j-1}},y_{t_{j+1}}) \\ &\quad= \frac{1}{\sqrt{2\pi\Delta t(\sigma_{t_j}^2+\sigma_{t_{j+1}}^2)}}\,e^{-\frac{(y_{t_{j+1}}-y_{t_{j-1}})^2}{2\Delta t(\sigma_{t_j}^2+\sigma_{t_{j+1}}^2)}} \qquad (13.45)\end{aligned}$$

Thus, in (13.44) we can integrate out all y_{t_j} with the exception of y_T to obtain ($s^2 := \sum_{j=1}^{N_T} \sigma_{t_j}^2$)

$$\begin{aligned} \mathsf{E}_W\Big[F\big(\textstyle\int_0^T \sigma_t \, dx_t\big) \Big] &= \lim_{\Delta t \to 0} \int_{\mathbb{R}} F(y_T) \, \tfrac{1}{\sqrt{2\pi \Delta t \, s^2}} \, e^{-\frac{(y_T - y_{t_0})^2}{2\Delta t \, s^2}} \, dy_T \\ &= \int_{\mathbb{R}} F(y_T) \, \tfrac{1}{\sqrt{2\pi T \sigma_{\mathrm{imp},T}^2}} \, e^{-\frac{y_T^2}{2T\sigma_{\mathrm{imp},T}^2}} \, dy_T \\ &= \int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T} \sqrt{T} \, x\big) \, \tfrac{1}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2}} \, dx \end{aligned} \tag{13.46}$$

since

$$\Delta t \, s^2 \;=\; \Delta t \textstyle\sum_{j=1}^{N_T} \sigma_{t_j}^2 \;\overset{\Delta t \to 0}{\to}\; \int_0^T \sigma_t^2 \, dt \;=\; T \, \sigma_{\mathrm{imp},T}^2$$

This proves part (a). For part (b), we use formulae (13.33) and (13.34) of Theorem 13.4 to obtain

$$\begin{aligned} V_0 &= e^{-rT} \mathsf{E}_W\big[H(S_T^{(r)})\big] \\ &= e^{-rT} \int H\big(S_0 \, e^{\int_0^T \sigma_t \, dx_t + \int_0^T (r - \frac{\sigma_t^2}{2}) dt}\big) \, dW(\{x_t\}_{0<t\le T}) \end{aligned} \tag{13.47}$$

such that part (b) is an immediate consequence of (13.41) of part (a). ■

As an immediate consequence of Theorem 13.5, we are in a position to write down the Black-Scholes formulae for the fair value of call- and put-options in the time dependent Black-Scholes model:

Corrolary 13.6: Consider standard european call- and put-options with strike K and maturity T,

$$\begin{aligned} H_{\mathrm{call}}(S_T) &= \max\{S_T - K, 0\} \\ H_{\mathrm{put}}(S_T) &= \max\{K - S_T, 0\} \end{aligned}$$

Suppose that the underlying asset price dynamics is given by the time-dependent Black-Scholes model

$$dS_t / S_t \;=\; \mu_t \, dt + \sigma_t \, dx_t \tag{13.48}$$

with some deterministic drift function μ_t and volatility function σ_t. Define the implied volatility $\sigma_{\mathrm{imp},T}$ at maturity T through the formula

$$\sigma_{\mathrm{imp},T} \;=\; \Big\{ \tfrac{1}{T} \textstyle\int_0^T \sigma_t^2 \, dt \Big\}^{1/2} \tag{13.49}$$

Then the time zero fair values of calls and puts are given by

$$V_{\text{call},0}^{\text{BSTD}} = S_0\, N(d_+) - K\, e^{-rT}\, N(d_-) \tag{13.50}$$

$$V_{\text{put},0}^{\text{BSTD}} = -S_0\, N(-d_+) + K\, e^{-rT}\, N(-d_-) \tag{13.51}$$

where

$$d_\pm := \frac{\log \frac{S_0}{K} + (r \pm \frac{\sigma_{\text{imp},T}^2}{2})T}{\sigma_{\text{imp},T}\sqrt{T}} \tag{13.52}$$

Proof: Follows immediately from (13.43) of Theorem 13.5 and the calculations in the proof of Theorem 6.1 ■

Calibration to Market

Let $V_{\text{call},K,T}^{\text{market}}$ be an observed market price for a call option with strike K and maturity T. Let $V_{\text{call},K,T}^{\text{BS}}(\sigma)$ be the theoretical fair value when this option is priced in the time-independent Black-Scholes model with volatility parameter σ. Then the implied volatility σ_{imp} of this option is defined to be that constant volatility number which has to be put into the time-independent Black-Scholes model in order to reproduce the market price. That is,

$$V_{\text{call},K,T}^{\text{BS}}\big(\sigma_{\text{imp}}\big) \stackrel{!}{=} V_{\text{call},K,T}^{\text{market}} \tag{13.53}$$

If one looks at concrete market prices, one finds that σ_{imp} is in fact a function of K and T,

$$\sigma_{\text{imp}} = \sigma_{\text{imp}}(K,T) \tag{13.54}$$

With a time-dependent Black-Scholes model, we can take care of the T-dependence of the volatilities, but not of the K-dependence. From Corollary 13.6, we have

$$V_{\text{call},K,T}^{\text{BS}}\big(\sigma_{\text{imp},T}\big) = V_{\text{call},K,T}^{\text{BSTD}}\big(\{\sigma_t\}_{0\leq t\leq T}\big) \tag{13.55}$$

if the implied volatility for maturity T and the volatility function $\{\sigma_t\}_{0\leq t\leq T}$ are related through the equation

$$T\,\sigma_{\text{imp},T}^2 = \int_0^T \sigma_t^2\, dt\ . \tag{13.56}$$

Now let

$$V_{\text{call},K,T_1}^{\text{market}},\ V_{\text{call},K,T_2}^{\text{market}},\ \cdots\ ,\ V_{\text{call},K,T_m}^{\text{market}} \tag{13.57}$$

be a set of observed market prices (say, for strike $K = S_0$, for 'at the money' calls) and let

$$\sigma_{\mathrm{imp},T_1}\,,\ \sigma_{\mathrm{imp},T_2}\,,\ \cdots\ ,\ \sigma_{\mathrm{imp},T_m} \tag{13.58}$$

be the corresponding implied volatilities. Now, calibrating the time-dependent Black-Scholes model to these market quotes means that we have to determine the volatility function σ_t of the time-dependent Black-Scholes model in such a way that the equation

$$T_k\,\sigma^2_{\mathrm{imp},T_k} \;=\; \int_0^{T_k} \sigma_t^2\,dt \tag{13.59}$$

is fulfilled for all observed maturities $T_1, \cdots, T_m$. From (13.59) we get

$$T_k\,\sigma^2_{\mathrm{imp},T_k} - T_{k-1}\,\sigma^2_{\mathrm{imp},T_{k-1}} \;=\; \int_{T_{k-1}}^{T_k} \sigma_t^2\,dt \tag{13.60}$$

Thus, if we let σ_t be a piecewise constant function, being equal to a constant σ_k on the intervalls (T_{k-1}, T_k), then we get from (13.60)

$$\sigma_k^2 \;=\; \frac{T_k\,\sigma^2_{\mathrm{imp},T_k} - T_{k-1}\,\sigma^2_{\mathrm{imp},T_{k-1}}}{T_k - T_{k-1}} \tag{13.61}$$

The process of choosing the σ_k according to (13.61) when the $\sigma_{\mathrm{imp},T_k}$ are given by market quotes is called 'calibrating the time-dependent Black-Scholes model to the market'.