Chapter 22

Stochastic Volatility Models

The basic assumption of the Black-Scholes model is that the price process S_t is given by a geometric Brownian motion,

$$dS_t = \mu_t S_t\, dt + \sigma S_t\, dx_t \tag{22.1}$$

with constant volatility σ. Now we drop the assumption of constant volatility and allow for a stochastic volatility. We assume that the (real, physical) price process is a solution of the SDE system

$$dS_t = \mu_t S_t\, dt + \sqrt{\nu_t}\, S_t\, dB_t^1 \tag{22.2}$$
$$d\nu_t = \alpha(S_t, \nu_t, t)\, dt + \beta(S_t, \nu_t, t)\sqrt{\nu_t}\, dB_t^2 \tag{22.3}$$

where α and β are some functions and dB_t^1 and dB_t^2 are two Brownian motions with correlation $\rho \in (-1, 1)$,

$$dB_t^1\, dB_t^2 = \rho\, dt \tag{22.4}$$

In terms of two uncorrelated Brownian motions dW_t^1 and dW_t^2, $dW_t^1 \cdot dW_t^2 = 0$, we can write

$$dB_t^1 = dW_t^1 \tag{22.5}$$
$$dB_t^2 = \rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2 \tag{22.6}$$

such that the system (22.2,22.3) can be rewritten as

$$d\begin{pmatrix} S \\ \nu \end{pmatrix} = \begin{pmatrix} \mu S \\ \alpha \end{pmatrix} dt + \begin{pmatrix} \sqrt{\nu}\, S & 0 \\ \beta\rho\sqrt{\nu} & \beta\sqrt{1-\rho^2}\,\sqrt{\nu} \end{pmatrix} \begin{pmatrix} dW^1 \\ dW^2 \end{pmatrix} \tag{22.7}$$

If we choose a Cox Ingersol Ross process for the volatility,

$$\alpha(S_t, \nu_t, t) = \lambda(\bar{\nu} - \nu_t) \tag{22.8}$$
$$\beta(S_t, \nu_t, t) = \eta \tag{22.9}$$

with constants λ, $\bar{\nu}$ and η, we obtain the Heston model which is explicitely solvable. It will be discussed in the next chapter.

Pricing and Hedging in Stochastic Volatility Models

Since the system (22.7) involves two sources of randomness, dW^1 and dW^2, we cannot exact replicate by using only the risky asset S as a hedging instrument. The model would be complete if we could use S and the volatility ν for hedging, but the volatility itself is not a tradable asset. However, there are liquid options which depend on the volatility. Thus, for our replicating portfolio we use S and some option with value $C(S,\nu)$. If we set up a selffinancing strategy which holds

$$\delta_{j-1} \text{ stocks at the beginning of time } j \text{ and } \delta_j \text{ stocks at the end of time } j \tag{22.10}$$

$$\gamma_{j-1} \text{ options at the beginning of time } j \text{ and } \gamma_j \text{ options at the end of time } j \tag{22.11}$$

then the same analysis as in lecture 1 shows that the discounted portfolio value $v_k = R^{-k}V_k$ at time k is given by

$$v_k = v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) + \sum_{j=1}^{k} \gamma_{j-1}(c_j - c_{j-1}) \tag{22.12}$$

where, as usual, $s_j = R^{-j}S_j$ is the discounted stock price process and

$$c_j = R^{-j}C(S_j, \nu_j) \tag{22.13}$$

is the discounted option value. Mathematically, C is just some function of ν which has to change if the volatility ν changes. That is,

$$\partial C/\partial \nu \neq 0 \qquad \text{for all } S, \nu\,. \tag{22.14}$$

Then exact replication in a suitably defined discrete time approximation of (22.7) is possible and v_k is the discounted price of the replicated contigent claim. In a similar way as for the Black-Scholes case, we can derive a PDE for $v_k \equiv v_{t_k} \equiv v_t$ in the continuous time limit. From (22.12) we get

$$dv = \delta\, ds + \gamma\, dc \tag{22.15}$$

or

$$dV - rV\, dt = \delta(dS - rS\, dt) + \gamma(dC - rC\, dt) \tag{22.16}$$

Abbreviating $V_t = \partial V/\partial t$, $V_{SS} = \partial^2 V/(\partial S)^2$ etc., we have

$$\begin{aligned} dV &= V_t\,dt + V_S\,dS + \tfrac{1}{2}V_{SS}(dS)^2 + V_\nu d\nu + \tfrac{1}{2}V_{\nu\nu}(d\nu)^2 + V_{S\nu}dSd\nu \\ &= V_t\,dt + V_S\big(\mu S\,dt + \sqrt{\nu}S\,dW_t^1\big) + \tfrac{\nu}{2}S^2V_{SS}\,dt \\ &\quad + V_\nu\Big(\alpha\,dt + \beta\sqrt{\nu}\big(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^2\big)\Big) + \tfrac{\nu}{2}\beta^2V_{\nu\nu}\,dt \\ &\quad + V_{S\nu}\rho\nu S\beta\,dt \\ &= \Big(V_t + \mu SV_S + \tfrac{\nu}{2}S^2V_{SS} + \alpha V_\nu + \tfrac{\nu}{2}\beta^2V_{\nu\nu} + \rho\nu\beta SV_{S\nu}\Big)dt \\ &\quad + \sqrt{\nu}(SV_S + \beta\rho V_\nu)\,dW_t^1 + \beta\sqrt{\nu}\sqrt{1-\rho^2}V_\nu\,dW_t^2 \end{aligned} \tag{22.17}$$

where we used (22.7) in the second line. The same expression is obtained for dC,

$$\begin{aligned} dC &= \Big(C_t + \mu SC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu}\Big)dt \\ &\quad + \sqrt{\nu}(SC_S + \beta\rho C_\nu)\,dW_t^1 + \beta\sqrt{\nu}\sqrt{1-\rho^2}C_\nu\,dW_t^2 \end{aligned} \tag{22.18}$$

Substituting (22.17) and (22.18) into (22.16), we arrive at

$$\begin{aligned} &\Big(V_t + \mu SV_S + \tfrac{\nu}{2}S^2V_{SS} + \alpha V_\nu + \tfrac{\nu}{2}\beta^2V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV\Big)dt + \sqrt{\nu}(SV_S + \beta\rho V_\nu)\,dW_t^1 \\ &\qquad + \beta\sqrt{\nu}\sqrt{1-\rho^2}V_\nu\,dW_t^2 \;=\; \delta(\mu - r)S\,dt + \delta\sqrt{\nu}S\,dW_t^1 \\ &+\gamma\Big(C_t + \mu SC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC\Big)dt \\ &\qquad + \gamma\sqrt{\nu}(SC_S + \beta\rho C_\nu)\,dW_t^1 + \gamma\beta\sqrt{\nu}\sqrt{1-\rho^2}C_\nu\,dW_t^2 \end{aligned} \tag{22.19}$$

Comparing the coefficients of dW_t^2, we find

$$\gamma \;=\; \frac{V_\nu}{C_\nu} \tag{22.20}$$

Comparing the coefficients of dW_t^1, we get

$$SV_S + \beta\rho V_\nu \;=\; \delta S + \gamma(SC_S + \beta\rho C_\nu) \tag{22.21}$$

or

$$\delta \;=\; V_S - \gamma C_S \;=\; V_S - \tfrac{V_\nu}{C_\nu}C_S \tag{22.22}$$

Finally, by comparing the drift coefficients, we get the PDE for V:

$$\begin{aligned} &V_t + \mu SV_S + \tfrac{\nu}{2}S^2V_{SS} + \alpha V_\nu + \tfrac{\nu}{2}\beta^2V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV \;= \\ &\quad= \;\delta(\mu - r)S + \gamma\Big(C_t + \mu SC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC\Big) \\ &\quad\overset{(22.22)}{=}\; V_S(\mu - r)S - \gamma C_S(\mu - r)S \\ &\qquad + \gamma\Big(C_t + \mu SC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC\Big) \end{aligned} \tag{22.23}$$

which gives

$$\begin{aligned} V_t + rSV_S + \tfrac{\nu}{2}S^2V_{SS} + \alpha V_\nu + \tfrac{\nu}{2}\beta^2 V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV &= \\ &= \gamma\Big(C_t + rSC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC\Big) \\ &= \tfrac{V_\nu}{C_\nu}\Big(C_t + rSC_S + \tfrac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC\Big) \end{aligned} \tag{22.24}$$

or

$$\begin{aligned} &\frac{V_t + rSV_S + \frac{\nu}{2}S^2V_{SS} + \alpha V_\nu + \frac{\nu}{2}\beta^2 V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV}{V_\nu} \\ &\quad = \frac{C_t + rSC_S + \frac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \frac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC}{C_\nu} \end{aligned} \tag{22.25}$$

Now, the left hand side of (22.25) only depends on the value (and derivatives thereof) of the contingent claim which is to be replicated. This value has to be independent of the hedging instruments since otherwise there would be arbitrage opportunities. That is, whatever option C we take as a hedging instrument, the right hand side of (22.25) has to be some universal function $\phi(t, S, \nu)$ which is independent of C. In other words, if the real market would behave according to (22.7) and all options would be consistently priced, then the quotient

$$\tfrac{C_t + rSC_S + \frac{\nu}{2}S^2C_{SS} + \alpha C_\nu + \frac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC}{C_\nu} \;=:\; \phi(t,S,\nu) \quad \text{independent of } C \tag{22.26}$$

has to be the same for all options. The PDE for V then reads

$$V_t + rSV_S + \tfrac{\nu}{2}S^2V_{SS} + (\alpha - \phi)V_\nu + \tfrac{\nu}{2}\beta^2 V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV \;=\; 0 \tag{22.27}$$

with final condition

$$V(T, S_T, \nu_T) \;=\; H(S_T) \tag{22.28}$$

if H denotes the payoff of the replicated contingent claim. In particular, the pricing equation does not depend on α since

$$\phi - \alpha \;=\; \tfrac{C_t + rSC_S + \frac{\nu}{2}S^2C_{SS} + \frac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta SC_{S\nu} - rC}{C_\nu} \;=:\; \tilde{\phi} \tag{22.29}$$

The Equivalent Martingale Measure

Let $h = e^{-rT}H$ or $h = R^{-N}H$ the discounted payoff of the contingent claim H which is replicated by the selffinancing strategy $(\delta_k, \gamma_k)_{0\le k\le N-1}$ at time N. Thus, by equation (22.12),

$$h \;=\; v_N \;=\; v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1}) + \sum_{j=1}^{k} \gamma_{j-1}(c_j - c_{j-1}) \tag{22.30}$$

or in continuous time

$$h = v_T = v_0 + \int_0^T \delta_t \, ds_t + \int_0^T \gamma_t \, dc_t \tag{22.31}$$

and we can write

$$v_0 = \mathsf{E}_{\tilde{W}}[\, h \,] \tag{22.32}$$

for any measure $\tilde{W}$ for which the processes (s_t) and (c_t) are martingales. To determine $\tilde{W}$, we use the Girsanov Theorem 14.2. We have

$$\begin{aligned} ds_t &= e^{-rt} dS_t - r s_t \, dt \\ &= (\mu - r) s_t \, dt + \sqrt{\nu}_t s_t \, dx_t^1 \end{aligned} \tag{22.33}$$

and

$$\begin{aligned} dc_t &= e^{-rt} dC_t - r c_t \, dt \qquad (22.34) \\ &= e^{-rt}\Big(C_t + \mu S C_S + \tfrac{\nu}{2} S^2 C_{SS} + \alpha C_\nu + \tfrac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta S C_{S\nu} - r C_t\Big) dt \\ &\qquad + \sqrt{\nu}(s c_s + \beta\rho c_\nu)\, dx_t^1 + \beta\sqrt{\nu}\sqrt{1-\rho^2} c_\nu \, dx_t^2 \\ &= \big((\mu - r) s c_s + \phi c_\nu\big) dt + \sqrt{\nu}(s c_s + \beta\rho c_\nu)\, dx_t^1 + \beta\sqrt{\nu}\sqrt{1-\rho^2} c_\nu \, dx_t^2 \end{aligned}$$

where $d\vec{x}_t = (dx_t^1, dx_t^2)$ is a two dimensional uncorrelated Brownian motion and ϕ is the function of (22.26). Also we used $e^{-rt} S C_S = e^{-rt} s C_s = s c_s$. In matrix notation, this reads

$$d\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} (\mu - r)s \\ (\mu - r) s c_s \phi c_\nu \end{pmatrix} dt + \sqrt{\nu} \begin{pmatrix} s & 0 \\ s c_s + \beta\rho c_\nu & \beta\sqrt{1-\rho^2} c_\nu \end{pmatrix} d\vec{x} \tag{22.35}$$

In order to eliminate the drift term, we make the substitution of variables $\{\vec{x}_t\} \to \{\vec{y}_t\}$ given by

$$\vec{y}_t = \vec{x}_t + \int_0^t \vec{u}_s \, ds \tag{22.36}$$

where

$$\begin{aligned} \vec{u}_t &= \frac{1}{\sqrt{\nu}_t} \begin{pmatrix} s & 0 \\ s c_s + \beta\rho c_\nu & \beta\sqrt{1-\rho^2} c_\nu \end{pmatrix}^{-1} \begin{pmatrix} (\mu - r)s \\ (\mu - r) s c_s + \phi c_\nu \end{pmatrix} \\ &= \frac{1}{\sqrt{\nu}_t} \begin{pmatrix} 1/s & 0 \\ -\frac{s c_s + \beta\rho c_\nu}{s\beta\sqrt{1-\rho^2} c_\nu} & \frac{1}{\beta\sqrt{1-\rho^2} c_\nu} \end{pmatrix} \begin{pmatrix} (\mu - r)s \\ (\mu - r) s c_s + \phi c_\nu \end{pmatrix} \\ &= \frac{1}{\sqrt{\nu}_t} \begin{pmatrix} \mu - r \\ -\frac{s c_s + \beta\rho c_\nu}{\beta\sqrt{1-\rho^2} c_\nu}(\mu - r) + \frac{1}{\beta\sqrt{1-\rho^2} c_\nu}\left[(\mu - r) s c_s + \phi c_\nu\right] \end{pmatrix} \\ &= \frac{1}{\sqrt{\nu}_t} \begin{pmatrix} \mu - r \\ -\frac{\rho}{\sqrt{1-\rho^2}}(\mu - r) + \frac{\phi}{\beta\sqrt{1-\rho^2}} \end{pmatrix} \\ &= \frac{1}{\sqrt{\nu}_t} \begin{pmatrix} \mu - r \\ \frac{\phi/\beta - \rho(\mu - r)}{\sqrt{1-\rho^2}} \end{pmatrix} \end{aligned} \tag{22.37}$$

Then, in terms of the new variables, s and c satisfy the SDE system

$$d\begin{pmatrix} s \\ c \end{pmatrix} = \sqrt{\nu}\begin{pmatrix} s & 0 \\ sc_s + \beta\rho c_\nu & \beta\sqrt{1-\rho^2}c_\nu \end{pmatrix} d\vec{y} \tag{22.38}$$

Thus, s and c are martingales with respect to the Wiener measure $dW(y)$ for the y-variables. In terms of the original x-variables, this measure is given by

$$dW_{(0,T]}\big(\{\vec{y}_t\}_{0<t\leq T}\big) = e^{-\int_0^T \vec{u}_s d\vec{x}_s - \frac{1}{2}\int_0^T \vec{u}_s^2 ds}\, dW_{(0,T]}\big(\{\vec{x}_t\}_{0<t\leq T}\big) \tag{22.39}$$

Hence, if $H(S_T) = e^{rT}h(S_T)$ denotes the payoff of some european option, then the price at time t is given by

$$\begin{aligned} V_t &= e^{-r(T-t)}\tilde{\mathsf{E}}\big[\, H\,|S_t\big] \\ &= e^{-r(T-t)}\int H\Big(S_t e^{\int_t^T \sqrt{\nu}_s dy_s^1 + \int_t^T (r-\frac{\nu_s}{2})ds}\Big)\, dW_{(t,T]}(y^1,y^2) \end{aligned} \tag{22.40}$$

Here we used the fact that

$$s_T = s_t\, e^{\int_t^T \sqrt{\nu}_s dy_s^1 - \int_t^T \frac{\nu_s}{2} ds} \tag{22.41}$$

is the solution of the SDE (22.38),

$$ds_u = \sqrt{\nu}_u\, dy_u^1\,, \qquad s_u\big|_{u=t} = s_t \tag{22.42}$$

Finally, what is the SDE for ν in terms of the new y variables? Again, we use the Girsanov Theorem 14.2. With respect to the x variables,

$$\begin{aligned} d\nu &= \alpha\, dt + \beta\sqrt{\nu}(\rho\, dx^1 + \sqrt{1-\rho^2}\, dx^2) \\ &= \alpha\, dt + \beta\sqrt{\nu}(\,\rho \quad \sqrt{1-\rho^2}\,)\begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix} \\ &\equiv \alpha\, dt + \sigma\, d\vec{x} \end{aligned} \tag{22.43}$$

If ν is considered as a function of the y variables, then, by the Girsanov theorem, it satisfies the SDE

$$d\nu = \tilde{\alpha}\, dt + \sigma\, d\vec{y} \tag{22.44}$$

where the new drift term is given by

$$\begin{aligned} \tilde{\alpha} &= \alpha - \sigma\vec{u} \\ &= \alpha - \beta\sqrt{\nu}\,(\,\rho \quad \sqrt{1-\rho^2}\,)\frac{1}{\sqrt{\nu}_t}\begin{pmatrix} \mu - r \\ \frac{\phi/\beta - \rho(\mu-r)}{\sqrt{1-\rho^2}} \end{pmatrix} \\ &= \alpha - \beta\big\{\rho(\mu-r) + \phi/\beta - \rho(\mu-r)\big\} \\ &= \alpha - \phi \end{aligned} \tag{22.45}$$

We summarize our results in the following

Theorem 22.1: Let the price process S_t of some risky asset be given by the solution of the SDE stochastic volatility system

$$dS_t = \mu_t\, S_t\, dt + \sqrt{\nu_t}\, S_t\, dB_t^1 \tag{22.46}$$

$$d\nu_t = \alpha(S_t, \nu_t, t)\, dt + \beta(S_t, \nu_t, t)\, \sqrt{\nu_t}\, dB_t^2 \tag{22.47}$$

where α and β are some functions and dB_t^1 and dB_t^2 are two Brownian motions with correlation $\rho \in (-1, 1)$, $dB_t^1\, dB_t^2 = \rho\, dt$. Let

$$H = H\big(\{S_t\}\big) \tag{22.48}$$

be the payoff of some (probably exotic) european option. Suppose constant interest rates r.

a) The price V_{t_0} at time t_0 for the contingent claim H is given by (dW being the usual 2-dimensional Wiener measure)

$$V_{t_0} = e^{-r(T-t_0)} \int H\Big(\big\{S_{t_0}\, e^{\int_{t_0}^t \sqrt{\nu}_s dy_s^1 + \int_{t_0}^t (r - \frac{\nu_s}{2}) ds}\big\}_{t_0 \le t \le T}\Big)\, dW_{(t_0,T]}(y^1, y^2) \tag{22.49}$$

where ν is a solution of the SDE

$$d\nu = (\alpha - \phi)\, dt + \beta \sqrt{\nu}\, \Big(\rho\, dy^1 + \sqrt{1-\rho^2}\, dy^2\Big) \tag{22.50}$$

and ϕ is the universal function given by (22.26). In particular, $\alpha - \phi$ does not depend on the (physical, real) drift α of (22.47) since

$$\phi - \alpha = \frac{C_t + rSC_S + \frac{\nu}{2} S^2 C_{SS} + \frac{\nu}{2}\beta^2 C_{\nu\nu} + \rho\nu\beta S C_{S\nu} - rC}{C_\nu} =: \tilde{\phi} \tag{22.51}$$

b) Let $C = C(S_t, \nu_t, t)$ be some liquid option which is used as a hedging instrument, besides S, and suppose $C_\nu \neq 0$ for all S, ν. Then a selffinancing replicating stategy is given by holding δ_t stocks S_t and γ_t options C_t at time t, where

$$\gamma = V_\nu / C_\nu \tag{22.52}$$

$$\delta = V_S - \gamma C_S = V_S - \tfrac{V_\nu}{C_\nu} C_S \tag{22.53}$$

and the discounted value $v_t = e^{-rt} V_t$ of the replicating portfolio at time t is given by

$$v_t = v_0 + \textstyle\int_0^t \delta_u\, ds_u + \int_0^t \gamma_u\, dc_u\,, \quad v_T = e^{-rT} H \tag{22.54}$$

c) If the payoff H depends only on S_T, $H = H(S_T)$, then the option price V_t is a solution of the PDE

$$V_t + rSV_S + \tfrac{\nu}{2}S^2V_{SS} - \tilde{\phi}V_\nu + \tfrac{\nu}{2}\beta^2V_{\nu\nu} + \rho\nu\beta SV_{S\nu} - rV \quad = \quad 0 \tag{22.55}$$

with final condition $V(T, S_T, \nu_T) = H(S_T)$ and $\tilde{\phi}$ given by (22.51).

Proof: It remains to prove that if V is a solution of (22.55) with final condition $V(T, S_T, \nu_T) = H(S_T)$ and γ and δ are choosen as in (22.52,22.53), then in fact

$$v_0 + \int_0^T \delta_t\, ds_t + \int_0^T \gamma_t\, dc_t \quad = \quad e^{-rT}H \tag{22.56}$$

is a replicating portfolio for the payoff H. For simplicity, we assume zero interest rates, $r = 0$. Then we have to prove

$$H \quad = \quad V_0 + \int_0^T \delta_t\, dS_t + \int_0^T \gamma_t\, dC_t \tag{22.57}$$

if (22.55) holds with final condition $V(T, S_T, \nu_T) = H(S_T)$ and if γ and δ are choosen as in (22.52,22.53). By use of the Ito formula, we have

$$\begin{aligned} H &= V(T, S_T, \nu_T) \;=\; V_0 + \int_0^T dV_t \qquad (22.58)\\ &= V_0 + \int_0^T \left\{V_t\, dt + V_S\, dS_t + \tfrac{\nu}{2}S^2V_{SS}\, dt + V_\nu\, d\nu_t + \tfrac{\nu}{2}\beta^2V_{\nu\nu}\, dt + \rho\nu\beta SV_{S\nu}\, dt\right\}\\ &= V_0 + \int_0^T \{(V_S - \gamma C_S)\, dS_t + \gamma dC_t\}\\ &\quad + \int_0^T \left\{\gamma C_S\, dS - \gamma dC_t + V_t\, dt + \tfrac{\nu}{2}S^2V_{SS}\, dt + V_\nu\, d\nu + \tfrac{\nu}{2}\beta^2V_{\nu\nu}\, dt + \rho\nu\beta SV_{S\nu}\, dt\right\} \end{aligned}$$

Since

$$\begin{aligned} \gamma C_S\, dS - \gamma dC_t + V_\nu\, d\nu &= \gamma C_S\, dS - \gamma\left[C_S\, dS + C_\nu\, d\nu + C_t\, dt\right] + V_\nu\, d\nu\\ &\quad -\gamma\left[\tfrac{\nu}{2}S^2C_{SS} + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu}\right] dt\\ &= -V_\nu \frac{C_t + \tfrac{\nu}{2}S^2C_{SS} + \tfrac{\nu}{2}\beta^2C_{\nu\nu} + \rho\nu\beta SC_{S\nu}}{C_\nu}\, dt\\ &= -V_\nu\tilde{\phi}\, dt \qquad (22.59) \end{aligned}$$

equation (22.58) becomes

$$\begin{aligned} H &= V_0 + \int_0^T \{\delta_t\, dS_t + \gamma_t\, dC_t\}\\ &\quad + \int_0^T \left\{-\tilde{\phi}\, V_\nu + V_t + \tfrac{\nu}{2}S^2V_{SS} + \tfrac{\nu}{2}\beta^2V_{\nu\nu} + \rho\nu\beta SV_{S\nu}\right\} dt \qquad (22.60)\\ &= V_0 + \int_0^T \{\delta_t\, dS_t + \gamma_t\, dC_t\} \end{aligned}$$

since the wavy brackets in (22.60) vanish by assumption on V. ■