Chapter 18

Pricing and Hedging in the Presence of Stochastic Volatility and Stochastic Interest Rates

The main objective of this chapter is to prove the Theorem 18.1 on page 149 below.

Suppose that the real, physical processes for the stock price S_t, for the variance ν_t and for the short rates r_t are given by the SDE's

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu\, dt + \sqrt{\nu_t}\, dB_t^S & (18.1) \\ d\nu_t &= \alpha(\nu_t, t)\, dt + \beta(\nu_t, t)\, dB_t^\nu & (18.2) \\ dr_t &= m(r_t, t)\, dt + \sigma\, dB_t^r & (18.3) \end{aligned}$$

where the interest rate volatility does not necessarily have to be a constant but may also depend on time and the short rate, $\sigma = \sigma(t, r_t)$. The Brownian motions dB^S, dB^ν and dB^r may have some correlations

$$\begin{aligned} dB^S \cdot dB^\nu &= \rho_{S,\nu}\, dt \\ dB^S \cdot dB^r &= \rho_{S,r}\, dt \\ dB^\nu \cdot dB^r &= \rho_{\nu,r}\, dt \end{aligned} \qquad (18.4)$$

Suppose we want to price some derivative with payoff $H = H(S_T)$. Since our stochastic model has 3 independent random factors, we need 3 linear independent instruments in order to replicate the payoff H. Besides the stock S_t, we choose some plain vanilla option $C = C(S, \nu, r, t)$ and some zero bond $P = P(r, t)$. We suppress the maturities of the plain vanilla option C and the zero bond P in the notation, they should be larger than T. Also, instead of H just depending on S_T, it could also depend on r_T and ν_T, but for notational simplicity, we omit r_T and ν_T. The exotic case $H = H\big(\{S_t, r_t\}_{0 \leq t \leq T}\big)$ is covered by part (c) of Theorem 18.1 on page 149 below.

Consider a self financing strategy which holds at time t_k δ_{t_k} stocks, η_{t_k} plain vanilla options and ρ_{t_k} zero bonds. Let V_0 be the initial investment, the costs to set up the self financing strategy which is the option price. The discounted portfolio value at time $t_N := T$, $v_{t_N} = DF_{t_N} V_{t_N}$, is given by (by an argument very similar to that given in the proof of Theorem 1.1 in the very first chapter)

$$v_{t_N} = v_0 + \sum_{k=1}^{N} \delta_{t_{k-1}}(s_{t_k} - s_{t_{k-1}}) + \sum_{k=1}^{N} \eta_{t_{k-1}}(c_{t_k} - c_{t_{k-1}}) + \sum_{k=1}^{N} \rho_{t_{k-1}}(p_{t_k} - p_{t_{k-1}}) \tag{18.5}$$

where the small letters denote discounted quantities. That is,

$$\begin{aligned} v_t &= DF_t\, V_t \\ s_t &= DF_t\, S_t \\ c_t &= DF_t\, C(S_t, \nu_t, r_t, t) \\ p_t &= DF_t\, P(r_t, t) \end{aligned} \tag{18.6}$$

with the discount factor

$$DF_t = e^{-\int_0^t r_u\, du} \tag{18.7}$$

Let us first determine the quantities δ, η and ρ which are needed to replicate the payoff. As a by-product, we will find that, if C and P are consistently priced within the model (18.1,18.2,18.3), the functions (18.30) and (18.35) below have to be some universal functions independent of the particular choice of P and C. This is of relevance since these functions will also show up in the equivalent martingale measure, the risk neutral pricing measure.

From equation (18.5), we obtain in continuous time

$$dv = \delta\, ds + \eta\, dc + \rho\, dp \tag{18.8}$$

which is equivalent to

$$dV - rV\, dt = \delta(dS - rS\, dt) + \eta(dC - rC\, dt) + \rho(dP - rP\, dt) \tag{18.9}$$

Since $C = C(S, \nu, r, t)$, we have

$$\begin{aligned}
dC &= C_S\, dS + C_\nu\, d\nu + C_r\, dr + \tfrac{1}{2}\big(C_{SS}\,(dS)^2 + C_{\nu\nu}\,(d\nu)^2 + C_{rr}(dr)^2\big) \\
&\qquad + C_{S\nu}\, dS\, d\nu + C_{Sr}\, dS\, dr + C_{\nu r}\, d\nu\, dr + C_t\, dt \\
&= C_S \sqrt{\nu} S\, dB^S + C_\nu \beta\, dB^\nu + C_r \sigma\, dB^r \\
&\quad + \Big(\mu S C_S + \alpha C_\nu + m C_r + \tfrac{1}{2}\big(S^2 \nu C_{SS} + \beta^2 C_{\nu\nu} + \sigma^2 C_{rr}\big) \\
&\quad + \beta \sqrt{\nu} S \rho_{S,\nu} C_{S\nu} + \sqrt{\nu} \sigma S \rho_{S,r} C_{Sr} + \beta \sigma \rho_{\nu,r} C_{\nu r} + C_t\Big) dt \\
&=: \mathcal{L}C\, dt + (\sqrt{\nu} S C_S, \beta C_\nu, \sigma C_r) \begin{pmatrix} dB^S \\ dB^\nu \\ dB^r \end{pmatrix} \\
&=: \mathcal{L}C\, dt + \langle \Sigma \nabla C, A dX \rangle \\
&= \mathcal{L}C\, dt + \langle A^T \Sigma \nabla C, dX \rangle \qquad (18.10)
\end{aligned}$$

where we defined the differential operator

$$\begin{aligned}
\mathcal{L}C &= \mu S C_S + \alpha C_\nu + m C_r + \tfrac{1}{2}\big(S^2 \nu C_{SS} + \beta^2 C_{\nu\nu} + \sigma^2 C_{rr}\big) \\
&\qquad + \beta \sqrt{\nu} S \rho_{S,\nu} C_{S\nu} + \sigma \sqrt{\nu} S \rho_{S,r} C_{Sr} + \sigma \beta \rho_{\nu,r} C_{\nu r} + C_t \qquad (18.11)
\end{aligned}$$

and the diagonal matrix

$$\Sigma = \begin{pmatrix} \sqrt{\nu} S & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \sigma \end{pmatrix} \qquad (18.12)$$

Furthermore A is a Cholesky root of the correlation matrix,

$$AA^T = \begin{pmatrix} 1 & \rho_{S,\nu} & \rho_{S,r} \\ \rho_{\nu,S} & 1 & \rho_{\nu,r} \\ \rho_{r,S} & \rho_{r,\nu} & 1 \end{pmatrix} \qquad (18.13)$$

and ∇C denotes the vector

$$\nabla C := \big(\tfrac{\partial C}{\partial S}, \tfrac{\partial C}{\partial \nu}, \tfrac{\partial C}{\partial r}\big) =: (C_S, C_\nu, C_r) \qquad (18.14)$$

Finally for vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, $\langle v, w \rangle := v_1 w_1 + v_2 w_2 + v_3 w_3$ denotes the scalar product and $dX = (dX_1, dX_2, dX_3)$ are 3 independent Brownian motions.

In the same way we obtain

$$dP = \mathcal{L}P\, dt + \langle A^T \Sigma \nabla P, dX \rangle \qquad (18.15)$$

where, since $P = P(r, t)$ does not depend on S and ν,

$$\mathcal{L}P = m P_r + \tfrac{\sigma^2}{2} P_{rr} + P_t \qquad (18.16)$$

and

$$\begin{aligned} dS &= \mu S\,dt + \sqrt{\nu} S\,dB^S \\ &= \mathcal{L} S\,dt + \langle A^T \Sigma \nabla S, dX \rangle \end{aligned} \tag{18.17}$$

Substituting this in (18.9) and abbreviating

$$\mathcal{L}_r C := \mathcal{L} C - rC \tag{18.18}$$

we get

$$\begin{aligned} \mathcal{L}_r V\,dt + \langle A^T \Sigma \nabla V, dX \rangle &= \delta\big(\mathcal{L}_r S dt + \langle A^T \Sigma \nabla S, dX \rangle\big) + \eta\big(\mathcal{L}_r C dt + \langle A^T \Sigma \nabla C, dX \rangle\big) \\ &\quad + \rho\big(\mathcal{L}_r P dt + \langle A^T \Sigma \nabla P, dX \rangle\big) \end{aligned} \tag{18.19}$$

This equation can only be fulfilled if the coefficients of dt, dX_1, dX_2 and dX_3 coincide. Thus we obtain:

$$\mathcal{L}_r V = \delta \mathcal{L}_r S + \eta \mathcal{L}_r C + \rho \mathcal{L}_r P \tag{18.20}$$

$$A^T \Sigma \nabla V = \delta A^T \Sigma \nabla S + \eta A^T \Sigma \nabla C + \rho A^T \Sigma \nabla P \tag{18.21}$$

Equation (18.21) is equivalent to

$$\nabla V = \delta \nabla S + \eta \nabla C + \rho \nabla P \tag{18.22}$$

This equation simply means that the whole portfolio consisting of the sold derivative plus the hedge position should be delta, vega and rho neutral. More explicitly,

$$\begin{aligned} V_S &= \delta + \eta C_S \\ V_\nu &= \eta C_\nu \\ V_r &= \eta C_r + \rho P_r \end{aligned} \tag{18.23}$$

which gives

$$\eta = \tfrac{V_\nu}{C_\nu} \tag{18.24}$$

$$\delta = V_S - \eta C_S \tag{18.25}$$

$$\rho = \tfrac{V_r}{P_r} - \eta \tfrac{C_r}{P_r} \tag{18.26}$$

Substituting these values in equation (18.20), we obtain

$$\mathcal{L}_r V = (V_S - \eta C_S)\mathcal{L}_r S + \eta \mathcal{L}_r C + V_r \tfrac{\mathcal{L}_r P}{P_r} - \eta C_r \tfrac{\mathcal{L}_r P}{P_r} \tag{18.27}$$

or, using $\mathcal{L}_r S = (\mu - r)S$,

$$\mathcal{L}_r V - (\mu - r) S V_S - V_r \tfrac{\mathcal{L}_r P}{P_r} = \eta \left(\mathcal{L}_r C - (\mu - r) S C_S - C_r \tfrac{\mathcal{L}_r P}{P_r} \right) \tag{18.28}$$

Now we have

$$\begin{aligned}
\frac{\mathcal{L}_r P}{P_r} &\overset{(18.16)}{=} \frac{mP_r + \frac{\sigma^2}{2}P_{rr} + P_t - rP}{P_r} \\
&= m + \frac{\frac{\sigma^2}{2}P_{rr} + P_t - rP}{P_r} \\
&=: m - \tilde{m} \qquad (18.29)
\end{aligned}$$

where we defined the function

$$\tilde{m}(r,t) := -\frac{\frac{\sigma^2}{2}P_{rr} + P_t - rP}{P_r} \qquad (18.30)$$

Furthermore, using (18.11) and (18.18), the left hand side of (18.28) can be rewritten as

$$\begin{aligned}
\mathcal{L}_r V - (\mu - r)SV_S - V_r \tfrac{\mathcal{L}_r P}{P_r} &= \mathcal{L}_r V - (\mu - r)SV_S - (m - \tilde{m})V_r \\
&= \mu SV_S + \alpha V_\nu + mV_r + \tfrac{1}{2}\big(S^2\nu V_{SS} + \beta^2 V_{\nu\nu} + \sigma^2 V_{rr}\big) \\
&\quad + \beta\sqrt{\nu}S\rho_{S,\nu}V_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}V_{Sr} + \sigma\beta\rho_{\nu,r}V_{\nu r} + V_t - rV \\
&\quad - (\mu - r)SV_S - (m - \tilde{m})V_r \\
&= rSV_S + \alpha V_\nu + \tilde{m}V_r + \tfrac{1}{2}\big(S^2\nu V_{SS} + \beta^2 V_{\nu\nu} + \sigma^2 V_{rr}\big) \\
&\quad + \beta\sqrt{\nu}S\rho_{S,\nu}V_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}V_{Sr} + \sigma\beta\rho_{\nu,r}V_{\nu r} + V_t - rV \\
&=: \tilde{\mathcal{L}}_r V \qquad (18.31)
\end{aligned}$$

Observe that $\tilde{\mathcal{L}}_r$ differs from $\mathcal{L}_r$ only through the substitutions

$$\begin{aligned}
\mu &\to r \\
m &\to \tilde{m} \qquad (18.32)
\end{aligned}$$

Thus, from (18.28) and (18.31) we obtain, using $\eta = \frac{V_\nu}{C_\nu}$,

$$\frac{\tilde{\mathcal{L}}_r V}{V_\nu} = \frac{\tilde{\mathcal{L}}_r C}{C_\nu} \qquad (18.33)$$

where we write, in analogy to (18.29),

$$\begin{aligned}
\frac{\tilde{\mathcal{L}}_r C}{C_\nu} &\overset{(18.31)}{=} \alpha + \frac{rSC_S + \tilde{m}C_r + \frac{1}{2}\big(S^2\nu C_{SS} + \beta^2 C_{\nu\nu} + \sigma^2 C_{rr}\big)}{C_\nu} \\
&\quad + \frac{\beta\sqrt{\nu}S\rho_{S,\nu}C_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}C_{Sr} + \sigma\beta\rho_{\nu,r}C_{\nu r} + C_t - rC}{C_\nu} \\
&=: \alpha - \tilde{\alpha} \qquad (18.34)
\end{aligned}$$

with the definition

$$\tilde{\alpha} \;:=\; -\frac{rSC_S + \tilde{m}C_r + \frac{1}{2}\big(S^2\nu C_{SS} + \beta^2 C_{\nu\nu} + \sigma^2 C_{rr}\big)}{C_\nu} - \frac{\beta\sqrt{\nu}S\rho_{S,\nu}C_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}C_{Sr} + \sigma\beta\rho_{\nu,r}C_{\nu r} + C_t - rC}{C_\nu} \quad (18.35)$$

Now, since the price V of the derivative cannot depend on a particular hedging instrument, the functions $\tilde{m}$ and $\tilde{\alpha}$ have to be some universal functions independent of the particular choice of P and C. We will check this later for $\tilde{m}$ in case of the Vasicek model. The PDE which has to be satisfied by $V = V(S,\nu,r,t)$, with payoff $V(S_T,\nu_T,r_T,T) = H(S_T)$, then reads

$$V_t - rV + rSV_S + \tilde{m}V_r + \tilde{\alpha}V_\nu + \tfrac{1}{2}\big(S^2\nu V_{SS} + \beta^2 V_{\nu\nu} + \sigma^2 V_{rr}\big) + \beta\sqrt{\nu}S\rho_{S,\nu}V_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}V_{Sr} + \sigma\beta\rho_{\nu,r}V_{\nu r} \;=\; 0 \quad (18.36)$$

In terms of the differential operator $\mathcal{L}_r$ (18.11,18.18), this reads

$$\mathcal{L}_r V - (\mu - r)SV_S - (m - \tilde{m})V_r - (\alpha - \tilde{\alpha})V_\nu \;=\; 0 \quad (18.37)$$

Observe that all original drift terms from the real, physical processes for S, ν and r, namely μ, α and m, have all disappeared. They are substituted by the new drift terms

$$\begin{aligned} \mu &\to r \\ m &\to \tilde{m} \\ \alpha &\to \tilde{\alpha} \end{aligned} \quad (18.38)$$

where $\tilde{m}$ and $\tilde{\alpha}$ are given by the universal functions (18.30) and (18.35). This substitution (18.38) is also found in the equivalent martingale measure which we compute now.

The Equivalent Martingale Measure

From equation (18.5), we have in the continuous time limit

$$DF_T H(S_T) \;=\; V_0 + \int_0^T \delta_t\, ds_t + \int_0^T \eta_t\, dc_t + \int_0^T \rho_t\, dp_t \quad (18.39)$$

where V_0 are the costs to set up the self financing strategy which replicates H, that is, the price of the option H. If all processes s_t, c_t and p_t are martingales with respect to some measure $\tilde{\mathsf{E}}[\,\cdot\,]$, we have $\tilde{\mathsf{E}}[ds_t] = \tilde{\mathsf{E}}[dc_t] = \tilde{\mathsf{E}}[dp_t] = 0$ such that

$$V_0 \;=\; \tilde{\mathsf{E}}[DF_T H(S_T)] \quad (18.40)$$

In the following we determine $\tilde{\mathsf{E}}[\,\cdot\,]$.

The real, physical processes for S, ν and r are given by (18.1,18.2,18.3),

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu\, dt + \sqrt{\nu_t}\, dB_t^S \\ d\nu_t &= \alpha(\nu_t, t)\, dt + \beta(\nu_t, t)\, dB_t^\nu \\ dr_t &= m(r_t, t)\, dt + \sigma\, dB_t^r \end{aligned}$$

with correlated Brownian motions dB^S, dB^ν and dB^r. Let A be a Cholesky root of the correlation matrix (18.13) and let $dX = (dX_1, dX_2, dX_3)$ be 3 independent Brownian motions such that, if $dB := (dB^S, dB^\nu, dB^r)$,

$$dB = A\, dX \tag{18.41}$$

Furthermore, let

$$\mathsf{E}[\,\cdot\,] = \int \cdot\; dW(\{X_{1,t}, X_{2,t}, X_{3,t}\}_{0<t\leq T}) \tag{18.42}$$

denote the expectation with respect to the Wiener measure for the 3 independent Brownian motions $X_t = (X_{1,t}, X_{2,t}, X_{3,t})$. In the following, we will make a Girsanov transformation ($\vec{u}_s$ to be determined)

$$Y_t := X_t + \textstyle\int_0^t \vec{u}_s\, ds \tag{18.43}$$

such that with respect to the new measure

$$dW(\{Y_{1,t}, Y_{2,t}, Y_{3,t}\}_{0<t\leq T}) = e^{-\int_0^T \vec{u}_s dX_s - \frac{1}{2}\int_0^T \vec{u}_s^2 ds}\, dW(\{X_{1,t}, X_{2,t}, X_{3,t}\}_{0<t\leq T}) \tag{18.44}$$

the processes s_t, c_t and p_t are all martingales.

We have $c_t = DF_t\, C(S_t, \nu_t, r_t, t)$ where the values S_t, ν_t and r_t which are substituted in C are the real, physical processes since the quantities which are relevant for the replicating strategy are of course the real, physical quantities. Thus, from (18.10,18.11,18.18), we get

$$\begin{aligned} dc_t &= DF_t\big(dC - rC\, dt\big) \\ &= DF_t\big(\mathcal{L}_r C\, dt + \langle \nabla C, \Sigma A dX\rangle\big) \end{aligned} \tag{18.45}$$

with similar equations for dp_t and ds_t. Collecting all 3 equations, this can be rewritten as

$$\begin{pmatrix} ds \\ dc \\ dp \end{pmatrix} = DF \begin{pmatrix} (\mu - r)S \\ \mathcal{L}_r C \\ \mathcal{L}_r P \end{pmatrix} dt + DF \begin{pmatrix} 1 & 0 & 0 \\ C_S & C_\nu & C_r \\ 0 & 0 & P_r \end{pmatrix} \Sigma A dX \tag{18.46}$$

If we make the transformation (18.43) with the vector $\vec{u}$ given by

$$\vec{u}_t = A^{-1}\Sigma^{-1}\begin{pmatrix} 1 & 0 & 0 \\ C_S & C_\nu & C_r \\ 0 & 0 & P_r \end{pmatrix}^{-1}\begin{pmatrix} (\mu - r)S \\ \mathcal{L}_r C \\ \mathcal{L}_r P \end{pmatrix} \tag{18.47}$$

then in terms of the new Brownian motions $Y_t = (Y_{1,t}, Y_{2,t}, Y_{3,t})$,

$$\begin{pmatrix} ds \\ dc \\ dp \end{pmatrix} = DF\begin{pmatrix} \sqrt{\nu}S & 0 & 0 \\ \sqrt{\nu}SC_S & \beta C_\nu & \sigma C_r \\ 0 & 0 & \sigma P_r \end{pmatrix} AdY \tag{18.48}$$

Thus, s_t, c_t and p_t are martingales with respect to the Wiener measure $dW(\{Y_t\}_{0<t\leq T})$ for the Y Brownian motions. Finally, we have to determine the SDE's for S, ν and r in terms of the Y Brownian motions. In terms of X,

$$\begin{pmatrix} dS \\ d\nu \\ dr \end{pmatrix} = \begin{pmatrix} \mu S \\ \alpha \\ m \end{pmatrix} dt + \begin{pmatrix} \sqrt{\nu}S & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \sigma \end{pmatrix} AdX \tag{18.49}$$

Substituting $dX = dY - \vec{u}\,dt$ in (18.49), we get

$$\begin{pmatrix} dS \\ d\nu \\ dr \end{pmatrix} = \text{new drift} \times dt + \begin{pmatrix} \sqrt{\nu}S & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \sigma \end{pmatrix} AdY \tag{18.50}$$

where the new drift is given by

$$\begin{aligned}
\text{new drift} &= \begin{pmatrix} \mu S \\ \alpha \\ m \end{pmatrix} - \Sigma A A^{-1}\Sigma^{-1}\begin{pmatrix} 1 & 0 & 0 \\ C_S & C_\nu & C_r \\ 0 & 0 & P_r \end{pmatrix}^{-1}\begin{pmatrix} (\mu - r)S \\ \mathcal{L}_r C \\ \mathcal{L}_r P \end{pmatrix} \\
&= \begin{pmatrix} \mu S \\ \alpha \\ m \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ -C_S/C_\nu & 1/C_\nu & -C_r/(C_\nu P_r) \\ 0 & 0 & 1/P_r \end{pmatrix}\begin{pmatrix} (\mu - r)S \\ \mathcal{L}_r C \\ \mathcal{L}_r P \end{pmatrix} \\
&= \begin{pmatrix} rS \\ \alpha - \frac{1}{C_\nu}\left(-(\mu - r)SC_S + \mathcal{L}_r C - C_r\frac{\mathcal{L}_r P}{P_r}\right) \\ m - \frac{\mathcal{L}_r P}{P_r} \end{pmatrix} \\
&\overset{(18.29)}{=} \begin{pmatrix} rS \\ \alpha - \frac{1}{C_\nu}\left(\mathcal{L}_r C - (\mu - r)SC_S - (m - \tilde{m})C_r\right) \\ \tilde{m} \end{pmatrix} \\
&\overset{(18.31)}{=} \begin{pmatrix} rS \\ \alpha - \frac{\tilde{\mathcal{L}}_r C}{C_\nu} \\ \tilde{m} \end{pmatrix} \\
&\overset{(18.34)}{=} \begin{pmatrix} rS \\ \tilde{\alpha} \\ \tilde{m} \end{pmatrix}
\end{aligned} \tag{18.51}$$

We summarize our results in the following

Theorem 18.1: Suppose that the real world (not risk neutral) processes for some stock S_t, variance ν_t and short term interest rate r_t are given by

$$\begin{aligned} \tfrac{dS_t}{S_t} &= \mu\, dt + \sqrt{\nu_t}\, dB_t^S \\ d\nu_t &= \alpha(\nu_t, t)\, dt + \beta(\nu_t, t)\, dB_t^\nu \\ dr_t &= m(r_t, t)\, dt + \sigma\, dB_t^r \end{aligned} \tag{18.52}$$

with correlations

$$\begin{aligned} dB^S \cdot dB^\nu &= \rho_{S,\nu}\, dt \\ dB^S \cdot dB^r &= \rho_{S,r}\, dt \\ dB^\nu \cdot dB^r &= \rho_{\nu,r}\, dt \end{aligned} \tag{18.53}$$

Let $H = H(S_T)$ be the payoff of some derivative which is to be priced and let

$$v_t = v_0 + \int_0^t \delta_u\, ds_u + \int_0^t \eta_u\, dc_u + \int_0^t \rho_u\, dp_u \tag{18.54}$$

be the discounted time t value of a self financing strategy which holds at time u δ_u stocks, η_u plain vanilla options $C(S_u, \nu_u, r_u, u)$ and ρ_u zero bonds $P(r_u, u)$. Then the following statements hold:

a) If the hedge instruments P and C are consistently priced in the model (18.52), then the functions

$$\tilde{m}(r,t) := -\frac{\frac{\sigma^2}{2} P_{rr} + P_t - rP}{P_r} \tag{18.55}$$

$$\begin{aligned} \tilde{\alpha}(S,\nu,r,t) := & -\frac{rSC_S + \tilde{m}C_r + \frac{1}{2}(S^2\nu C_{SS} + \beta^2 C_{\nu\nu} + \sigma^2 C_{rr})}{C_\nu} \\ & -\frac{\beta\sqrt{\nu}S\rho_{S,\nu}C_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}C_{Sr} + \sigma\beta\rho_{\nu,r}C_{\nu r} + C_t - rC}{C_\nu} \end{aligned} \tag{18.56}$$

have to be some universal functions independent of the particular choice of P and C.

b) Define the differential operator (for some function $V = V(S, \nu, r, t)$)

$$\begin{aligned} \mathcal{L}_{\text{risk neutral}} V := \; & rSV_S + \tilde{\alpha}V_\nu + \tilde{m}V_r + \tfrac{1}{2}\big(S^2\nu V_{SS} + \beta^2 V_{\nu\nu} + \sigma^2 V_{rr}\big) \\ & + \beta\sqrt{\nu}S\rho_{S,\nu}V_{S\nu} + \sigma\sqrt{\nu}S\rho_{S,r}V_{Sr} + \sigma\beta\rho_{\nu,r}V_{\nu r} + V_t - rV \end{aligned} \tag{18.57}$$

Suppose that V is a solution of the PDE

$$\begin{aligned} \mathcal{L}_{\text{risk neutral}} V &= 0 \\ V(S,\nu,r,T) &= H(S_T) \end{aligned} \tag{18.58}$$

and define

$$\begin{aligned} \eta &:= \tfrac{V_\nu}{C_\nu} && (18.59)\\ \delta &:= V_S - \eta C_S && (18.60)\\ \rho &:= \tfrac{V_r}{P_r} - \eta \tfrac{C_r}{P_r} && (18.61) \end{aligned}$$

Then (18.54) is in fact a replicating strategy for H. That is,

$$V(S,\nu,r,0) + \int_0^T \delta_t \, ds_t + \int_0^T \eta_t \, dc_t + \int_0^T \rho_t \, dp_t \;=\; e^{-\int_0^T r_t dt} \, H(S_T) \qquad (18.62)$$

where

$$\begin{aligned} s_t &= e^{-\int_0^T r_t dt} \, S_t \\ c_t &= e^{-\int_0^T r_t dt} \, C(S_t,\nu_t,r_t,t) \qquad (18.63)\\ p_t &= e^{-\int_0^T r_t dt} \, P(r_t,t) \end{aligned}$$

and (18.62) holds for all real world processes S_t, ν_t and r_t which are given by (18.52) (and which are to be substituted on the right hand side of (18.63)). In particular, the option price

$$\text{option price} \;=\; V(S,\nu,r,0) \qquad (18.64)$$

given by the solution of (18.58,18.57), is independent of μ, m and α but only depends on $\tilde{m}, \tilde{\alpha}$ and the vol and correlation parameters.

c) Let

$$H = H\big(\{S_t, r_t\}_{0\le t\le T}\big) \qquad (18.65)$$

be the payoff of some exotic option. Then the price of this option is given by

$$\text{price}(H) \;=\; \tilde{\mathsf{E}}\Big[e^{-\int_0^T \tilde{r}_t dt} \, H\big(\{\tilde{S}_t, \tilde{r}_t\}_{0\le t\le T}\big)\Big] \qquad (18.66)$$

where $(\tilde{S}_t, \tilde{\nu}_t, \tilde{r}_t)$ are given by the risk neutral SDE system

$$\begin{aligned} \tfrac{d\tilde{S}_t}{\tilde{S}_t} &= \tilde{r}_t \, dt + \sqrt{\tilde{\nu}_t} \, d\tilde{B}_t^S \\ d\tilde{\nu}_t &= \tilde{\alpha}(\tilde{\nu}_t,t) \, dt + \beta(\tilde{\nu}_t,t) \, d\tilde{B}_t^\nu \qquad (18.67)\\ d\tilde{r}_t &= \tilde{m}(\tilde{r}_t,t) \, dt + \sigma \, d\tilde{B}_t^r \end{aligned}$$

with correlated Brownian motions

$$\begin{aligned} d\tilde{B}^S \cdot d\tilde{B}^\nu &= \rho_{S,\nu} \, dt \\ d\tilde{B}^S \cdot d\tilde{B}^r &= \rho_{S,r} \, dt \qquad (18.68)\\ d\tilde{B}^\nu \cdot d\tilde{B}^r &= \rho_{\nu,r} \, dt \end{aligned}$$

Here $\tilde{m}$ and $\tilde{\alpha}$ are the universal functions (18.55,18.56).

Proof: It remains to prove that with the definition (18.59,18.60,18.61) of δ, η and ρ, the self financing strategy (18.54) in fact replicates H. That is, (18.62) holds. To this end, define the differential operator

$$\begin{aligned}\mathcal{L}_{\text{real world}} V &:= \mu S V_S + \alpha V_\nu + m V_r + \tfrac{1}{2}\big(S^2 \nu V_{SS} + \beta^2 V_{\nu\nu} + \sigma^2 V_{rr}\big) \\ &\qquad + \beta\sqrt{\nu} S \rho_{S,\nu} V_{S\nu} + \sigma\sqrt{\nu} S \rho_{S,r} V_{Sr} + \sigma\beta\rho_{\nu,r} V_{\nu r} + V_t - rV \\ &= \mathcal{L}_{\text{risk neutral}} V + \big\langle (\mu S - rS, \alpha - \tilde{\alpha}, m - \tilde{m}), \nabla V \big\rangle \end{aligned} \tag{18.69}$$

such that, as in (18.10),

$$\begin{aligned} dV &:= \lim_{\Delta t \to 0} \big(V(S_{t+\Delta t}, \nu_{t+\Delta t}, r_{t+\Delta t}, t + \Delta t) \; - \; V(S_t, \nu_t, r_t, t) \big) \\ &\overset{\substack{S_t,\nu_t,r_t \\ \text{real world} \\ \text{processes}}}{=} (\mathcal{L}_{\text{real world}} V + rV) dt + \langle \nabla V, \Sigma\, dB \rangle \end{aligned} \tag{18.70}$$

By assumption, V is a solution of the PDE (18.58), $\mathcal{L}_{\text{risk neutral}} V = 0$, which means

$$\mathcal{L}_{\text{real world}} V \;=\; \big\langle (\mu S - rS, \alpha - \tilde{\alpha}, m - \tilde{m}), \nabla V \big\rangle \tag{18.71}$$

Furthermore, because of (18.55,18.56) we also have

$$\mathcal{L}_{\text{risk neutral}} C \;=\; 0 \tag{18.72}$$

$$\mathcal{L}_{\text{risk neutral}} P \;=\; 0 \tag{18.73}$$

and by direct computation

$$\mathcal{L}_{\text{risk neutral}} S \;=\; 0 \tag{18.74}$$

Thus,

$$\mathcal{L}_{\text{real world}} S \;=\; \big\langle (\mu S - rS, \alpha - \tilde{\alpha}, m - \tilde{m}), \nabla S \big\rangle \tag{18.75}$$

$$\mathcal{L}_{\text{real world}} C \;=\; \big\langle (\mu S - rS, \alpha - \tilde{\alpha}, m - \tilde{m}), \nabla C \big\rangle \tag{18.76}$$

$$\mathcal{L}_{\text{real world}} P \;=\; \big\langle (\mu S - rS, \alpha - \tilde{\alpha}, m - \tilde{m}), \nabla P \big\rangle \tag{18.77}$$

Since δ, η and ρ are determined such that equation (18.22) holds,

$$\nabla V \;=\; \delta \nabla S + \eta \nabla C + \rho \nabla P \tag{18.78}$$

we also have

$$\mathcal{L}_{\text{real world}} V \;=\; \delta\, \mathcal{L}_{\text{real world}} S + \eta\, \mathcal{L}_{\text{real world}} C + \rho\, \mathcal{L}_{\text{real world}} P \tag{18.79}$$

which gives, using (18.70) and (18.78) again,

$$dV - rV dt \;=\; \delta\,(dS - rSdt) + \eta\,(dC - rCdt) + \rho\,(dP - rPdt) \tag{18.80}$$

or

$$dv \quad = \quad \delta\, ds + \eta\, dc + \rho\, dp \tag{18.81}$$

Thus, if

$$v(S_t, \nu_t, r_t, t) \quad := \quad e^{-\int_0^t r_u du} V(S_t, \nu_t, r_t, t) \tag{18.82}$$

then, abbreviating $V_0 = V(S_0, \nu_0, r_0, 0)$, we have

$$\begin{aligned} e^{-\int_0^T r_u du} H(S_T) - V_0 \quad &\overset{(18.58)}{=} \quad e^{-\int_0^T r_u du} V(S_T, \nu_T, r_T, T) - V_0 \\ &= \quad v(S_T, \nu_T, r_T, T) - v(S_0, \nu_0, r_0, 0) \\ &= \quad \int_0^T dv \\ &\overset{(18.81)}{=} \quad \int_0^T \delta\, ds + \int_0^T \eta\, dc + \int_0^T \rho\, dp \end{aligned} \tag{18.83}$$

which proves (18.62) ■