# Chapter 14

## The Multi-Underlying Black-Scholes Model and Correlation

So far we have discussed single asset options, the payoff function depended only on one underlying. Now we want to allow multiple underlyings. A standard example would be a simple basket call option with payoff

$$H_{\text{basket call}}(S_{1,T}, \cdots, S_{m,T}) \;=\; \max\left\{ \tfrac{1}{m}\textstyle\sum_{\ell=1}^{m} \tfrac{S_{\ell,T}}{S_{\ell,0}} - 1\,,\, 0 \right\} \tag{14.1}$$

Other examples could be options which measure relative performance. For example, a two-underlying option with underlyings $S_{1,t}$ and $S_{2,t}$ could simply pay out the relative performance

$$H(S_{1,T}, S_{2,T}) \;=\; \frac{S_{1,T}/S_{1,0}}{S_{2,T}/S_{2,0}} \tag{14.2}$$

If $S_1$ and $S_2$ have comparable volatilities, it is intuitively cleat that the fair value of the payoff (14.2) should be sensitive to correlation: If $S_1$ and $S_2$ move perfectly in sync, that is, if the returns $dS_1/S_1$ and $dS_2/S_2$ are highly correlated, it is clear that the payoff stays close to 1. On the other hand, if the returns $dS_1/S_1$ and $dS_2/S_2$ move in opposite directions, that is, if they are anti-correlated, the payoff can attain a broader range of values. In other words, correlation is a relevant model parameter and we recall some basic definitions and properties in the next section. The multi-underlying Black-Scholes model is then defined in section 14.2. For the payoff (14.2) we will actually find in example 14.7 below that its Black-Scholes time 0 price $V_0$ is given by

$$V_0 \;=\; e^{-rT}\, e^{(\sigma_2^2 - \sigma_1\sigma_2\rho)T} \tag{14.3}$$

if the underlyings $S_1$ and $S_2$ have time independent volatilities $\sigma_1$ and $\sigma_2$ and if underlying returns have the time independent correlation $\rho$. The fact that a multi asset option has a closed form solution for its price is quite exceptional: already the simple basket call with payoff (14.1) does not have a closed form solution in the multi-underlying Black-Scholes model. Therefore the standard calculation method for multi asset options is Monte Carlo.

## 14.1 Correlation

The correlation of two random variable $X$ and $Y$ is defined by

$$\mathsf{Corr}(X,Y) \;:=\; \frac{\mathsf{Cov}(X,Y)}{\{\,\mathsf{V}[X]\,\mathsf{V}[Y]\,\}^{1/2}} \tag{14.4}$$

where the covariance of $X$ and $Y$ is given by

$$\mathsf{Cov}(X,Y) \;:=\; \mathsf{E}\big[(X-\mathsf{E}[X])(Y-\mathsf{E}[Y])\big] \;=\; \mathsf{E}[XY]-\mathsf{E}[X]\mathsf{E}[Y] \tag{14.5}$$

In particular, $\mathsf{Corr}(X,X)=1$ and $\mathsf{Corr}(X,-X)=-1$. For random variables $X_1,...,X_m$, the covariance matrix

$$C \;=\; (c_{ij})_{1\le i,j\le m} \;:=\; \big(\mathsf{Cov}(X_i,X_j)\big)_{1\le i,j\le m} \tag{14.6}$$

is symmetric and positive definite (or more pricisely, at least positive semidefinite), since for arbitrary $\lambda_1,...,\lambda_m$

$$\sum_{i,j=1}^{d}\lambda_i\lambda_j c_{ij} \;=\; \mathsf{Cov}\Big(\sum_{i=1}^{m}\lambda_i X_i,\sum_{j=1}^{m}\lambda_j X_j\Big) \;=\; \mathsf{V}\Big[\sum_{i=1}^{m}\lambda_i X_i\Big] \;\ge\; 0 \tag{14.7}$$

By substituting $\lambda_i \to \lambda_i/\sqrt{\mathsf{V}[X_i]}$, the same conclusion holds for the correlation matrix

$$\rho \;=\; (\rho_{ij})_{1\le i,j\le m} \;:=\; \big(\mathsf{Corr}(X_i,X_j)\big)_{1\le i,j\le m} \tag{14.8}$$

In terms of the covariance matrix,

$$\rho_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}\;. \tag{14.9}$$

**Definition 14.1:** Let $C\in\mathbb{R}^{m\times m}$ be a symmetric matrix. Any matrix $A\in\mathbb{R}^{m\times m}$ which fulfills

$$AA^T \;=\; C \tag{14.10}$$

is called a Cholesky root of the matrix $C$.

**Example:** Let $-1\le\rho\le 1$ and

$$C \;=\; \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix} \tag{14.11}$$

then

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \tag{14.12}$$

is a Cholesky root for $C$.

**Lemma 14.2:** Every symmetric and positive semidefinite (that is, eigenvalues 0 can occur) matrix $C$ has a Cholesky root $A$. That is, for every symmetric and positive semidefinite $C \in \mathbb{R}^{m\times m}$ there is an $A \in \mathbb{R}^{m\times m}$ such that

$$AA^T = C \tag{14.13}$$

**Proof:** Since $C$ is symmetric, it can be diagonalized with an orthogonal matrix $V$,

$$C = VDV^T \tag{14.14}$$

with $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ and $\lambda_i \geq 0$ for all $1 \leq i \leq m$ and $V^{-1} = V^T$. Thus, if we put $\sqrt{D} = \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_m})$

$$\begin{aligned} C &= VDV^T \\ &= V\sqrt{D}\sqrt{D}V^T \\ &= V\sqrt{D}V^T V\sqrt{D}V^T \\ &= AA^T \end{aligned}$$

with $A = V\sqrt{D}V^T$. The $A$ we constructed here is also symmetric, $A = A^T$, but as the example above shows, this does not necessarily has to be the case. ■

The following theorem tells us how to construct random variables with a prescribed correlation from uncorrelated random variables.

**Theorem 14.3:** Let $Y_1, \cdots Y_m$ be $m$ independent random variables with mean 0 and variance 1. That is,

$$\mathsf{E}[Y_i] = 0\,, \quad \mathsf{V}[Y_i] = 1\,, \quad \mathsf{Corr}[Y_i, Y_j] = 0\,. \tag{14.15}$$

Let $C = (c_{ij})_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m}$ be a symmetric positive semidefinite matrix and let $\rho = (\rho_{ij})_{1\leq i,j\leq m}$ be the matrix with entries $\rho_{ij} = c_{ij}/\sqrt{c_{ii}c_{jj}}$. Let $A$ be a Cholesky root for $C$,

$$AA^T = C \tag{14.16}$$

Put $Y = (Y_1, \cdots Y_m)$ and let $X = (X_1, \cdots X_m)$ be given by

$$X = AY \tag{14.17}$$

or

$$\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = \begin{pmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, Y \rangle \\ \vdots \\ \langle \vec{a}_m, Y \rangle \end{pmatrix}$$

if we denote the row vectors of $A$ by $\vec{a}_i$ and let $\langle \cdot , \cdot \rangle$ denote the standard scalar product in $\mathbb{R}^m$. Then we have:

**a)** The $X_i$'s have variance $c_{ii}$ and correlation $\rho_{ij}$,

$$\mathsf{E}[X_i] = 0, \quad \mathsf{V}[X_i] = c_{ii} \quad \text{and} \quad \mathsf{Corr}[X_i, X_j] = \rho_{ij}. \tag{14.18}$$

That is, $C$ is the covariance matrix for the $(X_1, ..., X_m)$ and $\rho$ is their correlation matrix.

**b)** Suppose that the $Y_i$'s are normally distributed. That is, for any function $F = F(y_1, ..., y_m) : \mathbb{R}^m \to \mathbb{R}$ and with $\langle y, y \rangle = \sum_{i=1}^m y_i^2$,

$$\mathsf{E}[F(Y)] = \int_{\mathbb{R}^m} F(y)\, e^{-\frac{\langle y,y \rangle}{2}} \frac{d^m y}{\sqrt{(2\pi)^m}}. \tag{14.19}$$

Suppose further that all eigenvalues of $C$ are strictly positive, no zero eigenvalues. Then, with $X = AY$, we have

$$\mathsf{E}[F(X)] = \int_{\mathbb{R}^m} F(x)\, e^{-\frac{\langle x, C^{-1} x \rangle}{2}} \frac{d^m x}{\sqrt{(2\pi)^m \det C}} \tag{14.20}$$

**c)** Introduce the notation

$$C_m = \begin{pmatrix} & & & c_{1,m} \\ & C_{m-1} & & \vdots \\ & & & c_{m-1,m} \\ c_{m,1} & \cdots & c_{m,m-1} & c_{m,m} \end{pmatrix} \tag{14.21}$$

with

$$C_{m-1} = (c_{ij})_{1 \le i,j \le m-1} \in \mathbb{R}^{(m-1)\times(m-1)}.$$

Then, for $x = (x_1, ..., x_{m-1}, x_m) = (x', x_m)$ with $x' = (x_1, \cdots, x_{m-1}) \in \mathbb{R}^{m-1}$, we have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^m \det C_m}}\, e^{-\frac{\langle x, C_m^{-1} x \rangle}{2}}\, dx_m = \frac{1}{\sqrt{(2\pi)^{m-1} \det C_{m-1}}}\, e^{-\frac{\langle x', C_{m-1}^{-1} x' \rangle}{2}} \tag{14.22}$$

**Proof: a)** The equations $\mathsf{E}[X_i] = 0$ are an immediate consequence of $\mathsf{E}[Y_i] = 0$. Furthermore, by the assumption of independence of the $Y_i$'s and because of $\mathsf{V}[Y_i] = \mathsf{E}[Y_i^2] = 1$,

$$\mathsf{E}[Y_i Y_j] \;=\; \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \;=\; \delta_{i,j} \tag{14.23}$$

such that

$$\begin{aligned} \mathsf{E}[X_i X_j] &= \sum_{k,\ell=1}^{m} a_{ik} a_{j\ell}\, \mathsf{E}[Y_k Y_\ell] \\ &= \sum_{k,\ell=1}^{m} a_{ik} a_{j\ell}\, \delta_{k,\ell} \\ &= \sum_{k=1}^{m} a_{ik} a_{jk} \\ &= (AA^T)_{ij} \;=\; c_{ij} \end{aligned} \tag{14.24}$$

In particular, $\mathsf{V}[X_i] = \mathsf{E}[X_i^2] - \mathsf{E}[X_i]^2 = \mathsf{E}[X_i^2] = c_{ii}$ and

$$\mathsf{Corr}[X_i, X_j] \;=\; \mathsf{E}[X_i X_j] \;/\; \{\mathsf{V}[X_i]\mathsf{V}[X_j]\}^{1/2} \;=\; c_{ij}/\sqrt{c_{ii}c_{jj}} \;=\; \rho_{ij}$$

which proves part (a). Part (b) is obtained from the substitution of variables $x = Ay$,

$$\begin{aligned} \mathsf{E}[F(X)] &= \mathsf{E}[F(AY)] \\ &= \int_{\mathbb{R}^m} F(Ay)\, e^{-\frac{\langle y,y\rangle}{2}} \frac{d^m y}{\sqrt{(2\pi)^m}} \\ &\overset{y=A^{-1}x}{=} \int_{\mathbb{R}^m} F(x)\, e^{-\frac{\langle A^{-1}x, A^{-1}x\rangle}{2}} \frac{d^m(A^{-1}x)}{\sqrt{(2\pi)^m}} \\ &= \int_{\mathbb{R}^m} F(x)\, e^{-\frac{\langle x, (A^{-1})^T A^{-1}x\rangle}{2}}\, |\det(A^{-1})|\, \frac{d^m x}{\sqrt{(2\pi)^m}} \\ &= \int_{\mathbb{R}^m} F(x)\, e^{-\frac{\langle x, C^{-1}x\rangle}{2}} \frac{d^m x}{\sqrt{(2\pi)^m \det C}} \end{aligned} \tag{14.25}$$

since $|\det(A^{-1})| = 1/\sqrt{(\det A)^2} = 1/\sqrt{\det(AA^T)} = 1/\sqrt{\det C}$.

Finally, if we consider equation (14.25) above for some $F$ which depends actually only on $x_1, ..., x_{m-1}$, it is clear that part (c) should hold. However, let us prove equation (14.22) by explicit calculation:

**c)** Abbreviate $x = (x', x_m)$ with

$$\begin{aligned} x &= (x_1, ..., x_{m-1}, x_m) \in \mathbb{R}^m \\ x' &= (x_1, ..., x_{m-1}) \in \mathbb{R}^{m-1} \end{aligned}$$

Since we do not use the Cholesky root in this part of the proof, we use the notation $A$ for a different matrix: Write the $m\times m$ matrix $C_m^{-1}$ as a block matrix with one $(m-1)\times(m-1)$ block $A$, one $(m-1)\times 1$ block $B$, a $1\times(m-1)$ block $B^T$ and a $1\times 1$ number $D$,

$$C_m^{-1} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \tag{14.26}$$

Then the quadratic expression in the exponent on the left hand side of (14.22) can be written as

$$\begin{aligned} \langle x, C_m^{-1}x\rangle &= \Big\langle (x', x_m), \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \binom{x'}{x_m} \Big\rangle \\ &= \langle x', Ax'\rangle + 2(B^Tx')x_m + Dx_m^2 \end{aligned} \tag{14.27}$$

Thus we have to calculate the one dimensional gaussian integral

$$\begin{aligned} \int_{\mathbb{R}} e^{-\frac{1}{2}(\langle x',Ax'\rangle + 2(B^Tx')x_m + Dx_m^2)} dx_m &= \int_{\mathbb{R}} e^{-\frac{D}{2}\left(x_m^2 + 2\frac{(B^Tx')}{D}x_m\right) - \frac{1}{2}\langle x',Ax'\rangle} dx_m \\ &= \int_{\mathbb{R}} e^{-\frac{D}{2}\left(x_m + \frac{(B^Tx')}{D}\right)^2 + \frac{D}{2}\left(\frac{(B^Tx')}{D}\right)^2 - \frac{1}{2}\langle x',Ax'\rangle} dx_m \\ &= \int_{\mathbb{R}} e^{-\frac{D}{2}x^2} dx \;\times\; e^{\frac{1}{2}\frac{(B^Tx')^2}{D} - \frac{1}{2}\langle x',Ax'\rangle} \\ &= \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{D}} \;\times\; e^{-\frac{1}{2}\left(\langle x',Ax'\rangle - \frac{(B^Tx')^2}{D}\right)} \\ &= \sqrt{\frac{2\pi}{D}}\, e^{-\frac{1}{2}\left(\langle x',Ax'\rangle - \frac{(B^Tx')^2}{D}\right)} \end{aligned} \tag{14.28}$$

The exponent in (14.28) is again quadratic in $x'$ and can be rewritten as follows (we rewrite $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}^{m-1}}$ to make the dimensionality of the vectors in the scalar product more explicit)

$$\begin{aligned} \langle x', Ax'\rangle_{\mathbb{R}^{m-1}} - \tfrac{1}{D}\langle B^Tx', B^Tx'\rangle_{\mathbb{R}^1} &= \langle x', Ax'\rangle_{\mathbb{R}^{m-1}} - \tfrac{1}{D}\langle x', BB^Tx'\rangle_{\mathbb{R}^{m-1}} \\ &= \big\langle x', (A - BD^{-1}B^T)x'\big\rangle_{\mathbb{R}^{m-1}} \end{aligned} \tag{14.29}$$

Because of (14.26) (used in the second equality below, the first equality is just the definition of $C_m$ and $C_{m-1}$), we have

$$C_m = \begin{pmatrix} & & & c_{1,m} \\ & C_{m-1} & & \vdots \\ & & & c_{m-1,m} \\ c_{m,1} & \cdots & c_{m,m-1} & c_{m,m} \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} \tag{14.30}$$

Now use Lemma 10.2.2 of the extremely nice reference [11] (okay, okay..). It states that

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} \;=\; \begin{pmatrix} E & F \\ G & H \end{pmatrix} \tag{14.31}$$

where in particular $E$ is given by

$$E \;=\; (A - BD^{-1}B^T)^{-1} \tag{14.32}$$

On the other hand, we have $E = C_{m-1}$. Thus from (14.30) and (14.29) we get

$$\begin{aligned} \langle x', Ax'\rangle_{\mathbb{R}^{m-1}} - \tfrac{1}{D}\langle B^T x', B^T x'\rangle_{\mathbb{R}^1} &= \big\langle x', (A - BD^{-1}B^T)x'\big\rangle_{\mathbb{R}^{m-1}} \\ &= \langle x', E^{-1}x'\rangle_{\mathbb{R}^{m-1}} \;=\; \langle x' C_{m-1}^{-1} x'\rangle_{\mathbb{R}^{m-1}} \end{aligned} \tag{14.33}$$

and from (14.28) we obtain

$$\begin{aligned} \frac{1}{\sqrt{(2\pi)^m \det C_m}} \int_{\mathbb{R}} e^{-\frac{1}{2}\langle x, C_m^{-1} x\rangle} dx_m &= \frac{1}{\sqrt{(2\pi)^m \det C_m}} \sqrt{\tfrac{2\pi}{D}}\; e^{-\frac{1}{2}\left(\langle x', Ax'\rangle - \frac{(B^T x')^2}{D}\right)} \\ &= \frac{1}{\sqrt{(2\pi)^{m-1} D \det C_m}}\; e^{-\frac{1}{2}\langle x', C_{m-1}^{-1} x'\rangle} \end{aligned} \tag{14.34}$$

Thus it needs to be checked that $D \times \det C_m = \det C_{m-1}$. However, this follows immediately from Cramer's rule since $D = (C_m^{-1})_{m,m}$ and, by Cramer's rule, inverse matrix elements are given by quotients of subdeterminants or 'Adjunkten', in this case $\det C_{m-1}$, divided by the determinant of the whole matrix, in this case $\det C_m$. This proves part (c) and the theorem. ■

The multi-underlying Black-Scholes model is defined in terms of correlated Brownian motions. They are constructed in the following

**Lemma 14.4:** Let $y_t = (y_{1,t}, \cdots, y_{m,t})$ be a vector of $m$ uncorrelated Brownian motions given by

$$y_t \;=\; \lim_{\Delta t \to 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \vec{\phi}_k \tag{14.35}$$

with $\vec{\phi}_k = (\phi_{k,1}, \cdots, \phi_{k,m})$ and the $\phi_{k,\ell}$ being independent (in both dimensions $k$ and $\ell$) and normally distributed random numbers with mean 0 and variance 1. Let

$$\rho \;=\; \rho(t) \;=\; \big(\rho_{i,j}(t)\big)_{1 \le i,j \le m} \;\in\; \mathbb{R}^{m \times m} \tag{14.36}$$

be a (eventually time dependent) correlation matrix and let $A = A(t)$ be a Cholesky root for $\rho(t)$, that is, $AA^T = \rho$ or, if we denote the row vectors of $A$ by $\vec{a}_i = \vec{a}_i(t)$,

$$\langle \vec{a}_i(t), \vec{a}_j(t) \rangle = \rho_{i,j}(t) \tag{14.37}$$

Then, if we put $x_t = (x_{1,t}, \cdots, x_{m,t})$ with

$$x_{i,t} = \lim_{\Delta t \to 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \langle \vec{a}_i(t_k), \vec{\phi}_k \rangle \tag{14.38}$$

and $t_k = k\Delta t$, we have

$$dx_{i,t} dx_{j,t} = \rho_{i,j}(t)\, dt \tag{14.39}$$

where the above equation (14.39) is just the differential notation of the following more precise statement on the quadratic covariation of the correlated Brownian motions $x_{1,t}, ..., x_{m,t}$ (with $\Delta x_{i,t_k} = x_{i,t_k} - x_{i,t_{k-1}}$):

$$\lim_{\Delta t \to 0} \sum_{k=1}^{t/\Delta t} \Delta x_{i,t_k} \Delta x_{j,t_k} = \int_0^t \rho_{i,j}(u)\, du \tag{14.40}$$

**Proof:** For fixed $\Delta t$ we put

$$\begin{aligned} I_t &:= \sum_{k=1}^{t/\Delta t} \Delta x_{i,t_k} \Delta x_{j,t_k} \\ &= \Delta t \sum_{k=1}^{t/\Delta t} \langle \vec{a}_i(t_k), \vec{\phi}_k \rangle \langle \vec{a}_j(t_k), \vec{\phi}_k \rangle \end{aligned} \tag{14.41}$$

We calculate the mean and the variance of $I_t$. The main observation will be again (we used this reasoning already to show that the quadratic variation of a 1-dimensional Brownian motion is deterministic and equal to $t$) that the variance of $I_t$ converges to 0 as $\Delta t \to 0$ which means that $I_t$ itself converges to its mean in probability as a consequence of Tchebyscheff's inequality.

The mean of $I_t$ is given by

$$\begin{aligned}
\mathsf{E}[\,I_t\,] &= \Delta t \sum_{k=1}^{t/\Delta t} \mathsf{E}[\,\langle \vec{a}_i(t_k), \vec{\phi}_k\rangle\langle \vec{a}_j(t_k), \vec{\phi}_k\rangle\,] \\
&= \Delta t \sum_{k=1}^{t/\Delta t} \sum_{\ell,\ell'=1}^{m} a_{i,\ell}(t_k) a_{j,\ell'}(t_k)\, \mathsf{E}[\,\phi_{k,\ell}\phi_{k,\ell'}\,] \\
&= \Delta t \sum_{k=1}^{t/\Delta t} \sum_{\ell,\ell'=1}^{m} a_{i,\ell}(t_k) a_{j,\ell'}(t_k)\, \delta_{\ell,\ell'} \\
&= \Delta t \sum_{k=1}^{t/\Delta t} \langle \vec{a}_i(t_k), \vec{a}_j(t_k)\rangle \\
&\overset{(14.37)}{=} \Delta t \sum_{k=1}^{t/\Delta t} \rho_{i,j}(t_k) \;\overset{\Delta t \to 0}{\to}\; \int_0^t \rho_{i,j}(u)\, du
\end{aligned} \tag{14.42}$$

and, since $I_t$ is given by a sum of independent random variables, its variance converges to zero:

$$\begin{aligned}
\mathsf{V}[\,I_t\,] &= \mathsf{V}\Big[\Delta t \sum_{k=1}^{t/\Delta t} \langle \vec{a}_i(t_k), \vec{\phi}_k\rangle\langle \vec{a}_j(t_k), \vec{\phi}_k\rangle\Big] \\
&= \Delta t^2 \sum_{k=1}^{t/\Delta t} \mathsf{V}\big[\langle \vec{a}_i(t_k), \vec{\phi}_k\rangle\langle \vec{a}_j(t_k), \vec{\phi}_k\rangle\big] \\
&= \Delta t^2 \sum_{k=1}^{t/\Delta t} \mathsf{V}\big[\langle \vec{a}_i(t_k), \vec{\phi}_1\rangle\langle \vec{a}_j(t_k), \vec{\phi}_1\rangle\big] \\
&\approx \Delta t \; \underbrace{\int_0^t du\, \mathsf{V}\big[\langle \vec{a}_i(u), \vec{\phi}_1\rangle\langle \vec{a}_j(u), \vec{\phi}_1\rangle\big]}_{\text{some constant}} \;\overset{\Delta t \to 0}{\to}\; 0
\end{aligned} \tag{14.43}$$

From (14.42) and (14.43) we get the convergence in probability of $I_t$ to $\int_0^t \rho_{i,j}(u)du$. ■

## 14.2 The Multi-Underlying Black-Scholes Model

The time-dependent multi-underlying Black-Scholes model is given by $m$ underlyings $S_t = (S_{1,t}, \cdots, S_{m,t})$ with dynamics

$$\frac{dS_{i,t}}{S_{i,t}} = \mu_{i,t}\, dt + \sigma_{i,t}\, dx_{i,t} \tag{14.44}$$

with $m$ correlated Brownian motions $x_t = (x_{1,t}, \cdots, x_{m,t})$ with correlations

$$dx_{i,t}dx_{j,t} \;=\; \rho_{i,j}(t)\,dt \tag{14.45}$$

As in the single underlying case, equation (14.44) is solved by

$$S_{i,t} \;=\; S^{(\mu)}_{i,t} \;=\; S_{i,0}\,e^{\int_0^t \sigma_{i,u}dx_{i,u}+\int_0^t(\mu_{i,u}-\sigma^2_{i,u}/2)du} \tag{14.46}$$

In terms of some uncorrelated Brownian motions $y_t = (y_{1,t}, \cdots, y_{m,t})$, we can write

$$dx_t \;=\; A(t)dy_t \tag{14.47}$$

if $A(t)$ is a Cholesky root for the correlation matrix $\big(\rho_{i,j}(t)\big)$. If we denote the row vectors of $A(t)$ by $\vec{a}_i(t)$ again, (14.47) reads in components

$$dx_{i,t} \;=\; \vec{a}_i(t)dy_t \tag{14.48}$$

with the multiplication on the right hand side of (14.48) being a scalar product of $m$-dimensional vectors. Thus we have

$$S_{i,t} \;=\; S^{(\mu)}_{i,t} \;=\; S_{i,0}\,e^{\int_0^t \sigma_{i,u}\vec{a}_i(u)dy_u+\int_0^t(\mu_{i,u}-\sigma^2_{i,u}/2)du} \tag{14.49}$$

The following theorem is the analog of Theorem 13.4 now for the multi asset case. It states that also in the multi-underlying case, we still can calculate option prices by simply taking expectation values of the payoff function. Again, the drift parameters $\mu_{i,t}$ have to be substituted by the interest rate $r$.

**Theorem 14.5:** Let $S_t = (S_{1,t}, \cdots, S_{m,t}) = S^{(\mu)}_t$ abbreviate $m$ underlyings with correlated time dependent Black-Scholes dynamics given by (14.44,14.45) or (14.49). Let $H = H(\{S^{(\mu)}_t\})$ be some multi asset option payoff. Then the price $V_0 = V_0^{\text{BSTD\_MU}}$ of this multi asset option is given by

$$\begin{aligned} V_0^{\text{BSTD\_MU}} &= e^{-rT}\mathsf{E}_W\big[H(\{S^{(r)}_t\})\big] \\ &= e^{-rT}\int H\big(\{S^{(r)}_t(y)\}\big)\,dW(y) \end{aligned} \tag{14.50}$$

where $S^{(r)}_t(y) = \big(S^{(r)}_{1,t}(y), \cdots, S^{(r)}_{m,t}(y)\big)$ and

$$S^{(r)}_{i,t}(y) \;=\; S_{i,0}\,e^{\int_0^t \sigma_{i,u}\vec{a}_i(u)dy_u+\int_0^t(r-\sigma^2_{i,u}/2)du} \tag{14.51}$$

and the Wiener measure $dW(y)$ for the $m$-dimensional uncorrelated Brownian motion $y_t = (y_{1,t}, \cdots, y_{m,t})$ is given by ($\Delta y_{t_k} = y_{t_k} - y_{t_{k-1}}$ and the limit $\Delta t \to 0$, $N_T \to \infty$ to be taken)

$$dW(y) \;=\; \prod_{k=1}^{N_T} \frac{1}{\sqrt{[2\pi(t_k-t_{k-1})]^m}}\, e^{-\frac{1}{2(t_k-t_{k-1})}\langle \Delta y_{t_k}, \Delta y_{t_k}\rangle}\, d^m y_{t_k} \tag{14.52}$$

The proof is analog to the single asset case and we skip it. Instead of this, we will now consider the question of how the expectation (14.50) can be concretely calculated if the payoff $H$ depends only on the underlying values at maturity, $H = H(S_T)$. That is, we write down and prove the analog of Theorem 13.5 which reads as follows in the multi asset case:

**Theorem 14.6: a)** Let $y_t = (y_{1,t}, \cdots, y_{m,t})$ denote an $m$-dimensional uncorrelated Brownian motion, let $\sigma_{1,t}, ..., \sigma_{m,t}$ be some instantaneous volatilities and let $A(t)A(t)^T = (\rho_{i,j}(t))$ or, if we denote the row vectors of $A$ by $\vec{a}_i(t)$,

$$\vec{a}_i(t)\vec{a}_j(t) \;=\; \rho_{i,j}(t) \tag{14.53}$$

Then we have for any function $F : \mathbb{R}^m \to \mathbb{R}$

$$\int F\Big(\big\{\textstyle\int_0^T \sigma_{i,t}\vec{a}_i(t)dy_t\big\}_{1\le i\le m}\Big)\, dW(y) \;=\; \int_{\mathbb{R}^m} F(x_1, \cdots, x_m)\, \frac{e^{-\frac{1}{2}\langle x, C_T^{-1}x\rangle} d^m x}{\sqrt{(2\pi)^m \det C_T}} \tag{14.54}$$

$$\;=\; \int_{\mathbb{R}^m} F(A_T y)\, e^{-\frac{1}{2}\langle y,y\rangle}\, \frac{d^m y}{\sqrt{(2\pi)^m}} \tag{14.55}$$

with a covariance matrix

$$C_T \;=\; \Big(\textstyle\int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j}(t)dt\Big)_{1\le i,j\le m} \tag{14.56}$$

and $A_T$ being a Cholesky root for $C_T$, $A_T A_T^T = C_T$ (with the subscript $T$ denoting maturity $T$ and the superscript $T$ denoting matrix transposition).

**b)** Let $H = H(S_{1,T}, \cdots, S_{m,T})$ be the payoff of some non path-dependent multi asset option. Then its fair value $V_0 = V_0^{\mathrm{BSTD\_MU}}$ in the time-dependent multi-underlying Black-Scholes model is given by

$$V_0 \;=\; e^{-rT} \int_{\mathbb{R}^m} H\Big(\Big\{S_{i,0}\, e^{\sigma_{i,T}^{\mathrm{imp}}\sqrt{T}\,x_i + (r - \frac{(\sigma_{i,T}^{\mathrm{imp}})^2}{2})T}\Big\}_{1\le i\le m}\Big)\, \frac{e^{-\frac{1}{2}\langle x, \rho_{\mathrm{imp},T}^{-1}x\rangle} d^m x}{\sqrt{(2\pi)^m \det \rho_{\mathrm{imp},T}}} \tag{14.57}$$

$$\;=\; e^{-rT} \int_{\mathbb{R}^m} H\Big(\Big\{S_{i,0}\, e^{\sigma_{i,T}^{\mathrm{imp}}\sqrt{T}\,\langle \vec{a}_{\mathrm{imp},i}, y\rangle + (r - \frac{(\sigma_{i,T}^{\mathrm{imp}})^2}{2})T}\Big\}_{1\le i\le m}\Big)\, e^{-\frac{1}{2}\langle y,y\rangle}\, \frac{d^m y}{\sqrt{(2\pi)^m}} \tag{14.58}$$

with implied volatilities

$$\sigma_{i,T}^{\text{imp}} = \left\{ \tfrac{1}{T} \int_0^T \sigma_{i,t}^2 \, dt \right\}^{1/2} \tag{14.59}$$

and an implied correlation matrix $\rho_{\text{imp},T} = \left(\rho_{i,j}^{\text{imp}}(T)\right)_{1 \leq i,j \leq m}$ with entries

$$\rho_{i,j}^{\text{imp}}(T) := \frac{\int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j}(t)dt}{\left\{\int_0^T \sigma_{i,t}^2 dt \int_0^T \sigma_{j,t}^2 dt\right\}^{1/2}} \tag{14.60}$$

If we let $A_{\text{imp}}$ denote a Cholesky root of that matrix,

$$A_{\text{imp}} A_{\text{imp}}^T = \rho_{\text{imp},T} \tag{14.61}$$

then the vectors $\vec{a}_{\text{imp},i}$ showing up on the right hand side of (14.58) in the exponential function are the row vectors of that $A_{\text{imp}} \in \mathbb{R}^{m \times m}$.

**Proof: a)** Let

$$\begin{aligned} x_i &:= \int_0^T \sigma_{i,t} \vec{a}_i(t) dy_t \\ &= \lim_{\Delta t \to 0} \sum_{k=1}^{N_T} \sigma_{i,t_k} \vec{a}_i(t_k) \sqrt{\Delta t} \, \phi_k \end{aligned}$$

with $\phi_k = (\phi_{k,1}, \cdots, \phi_{k,m})$ being independent normal random numbers. In matrix notation, with $x = (x_1, \cdots, x_m)$ and $\sigma_{t_k} := \text{diag}(\sigma_{1,t_k}, \cdots, \sigma_{m,t_k}) \in \mathbb{R}^{m \times m}$,

$$x = \lim_{\Delta t \to 0} \sum_{k=1}^{N_T} \sigma_{t_k} A(t_k) \sqrt{\Delta t} \, \phi_k \tag{14.62}$$

with Wiener measure

$$dW(\phi) = \prod_{k=1}^{N_T} e^{-\frac{\langle \phi_k, \phi_k \rangle}{2}} \frac{d^m \phi_k}{\sqrt{(2\pi)^m}} \tag{14.63}$$

Now we make the substitution of variables $\phi_k \to \psi_k$ where

$$\begin{aligned} & \psi_k = \sigma_{t_k} A(t_k) \, \phi_k \\ \Leftrightarrow \quad & \phi_k = A(t_k)^{-1} \sigma_{t_k}^{-1} \psi_k \end{aligned} \tag{14.64}$$

which gives

$$x = \lim_{\Delta t \to 0} \sum_{k=1}^{N_T} \sqrt{\Delta t} \, \psi_k \tag{14.65}$$

with Wiener measure

$$
\begin{aligned}
dW(\psi) &= \prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle A(t_k)^{-1}\sigma_{t_k}^{-1}\psi_k, A(t_k)^{-1}\sigma_{t_k}^{-1}\psi_k\rangle} \frac{1}{|\det A(t_k)\det\sigma_{t_k}|}\frac{d^m\psi_k}{\sqrt{(2\pi)^m}} \\
&= \prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle \psi_k, \sigma_{t_k}^{-1}[A(t_k)A(t_k)^T]^{-1}\sigma_{t_k}^{-1}\psi_k\rangle} \frac{1}{\sqrt{\det[\sigma_{t_k}A(t_k)A(t_k)^T\sigma_{t_k}]}}\frac{d^m\psi_k}{\sqrt{(2\pi)^m}} \\
&= \prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle \psi_k, C(t_k)^{-1}\psi_k\rangle} \frac{1}{\sqrt{\det[C(t_k)]}}\frac{d^m\psi_k}{\sqrt{(2\pi)^m}} \qquad (14.66)
\end{aligned}
$$

where we put

$$
C(t_k) \;:=\; \sigma_{t_k}A(t_k)A(t_k)^T\sigma_{t_k} \;\in\; \mathbb{R}^{m\times m} \qquad (14.67)
$$

which has the matrix entries

$$
c_{i,j}(t_k) \;=\; \sigma_{i,t_k}\sigma_{j,t_k}\,\rho_{i,j}(t_k) \qquad (14.68)
$$

Now we transform back to the Brownian motion type integration variables by putting

$$
\begin{aligned}
x_{t_k} &:= \sqrt{\Delta t}\sum_{j=1}^{k}\psi_j \qquad (14.69) \\
\Leftrightarrow \qquad \psi_k &= \frac{x_{t_k}-x_{t_{k-1}}}{\sqrt{\Delta t}}
\end{aligned}
$$

and arrive at (the limit $\Delta t \to 0$ to be taken)

$$
\begin{aligned}
&\int F\Big(\big\{\textstyle\int_0^T \sigma_{i,t}\vec{a}_i(t)dy_t\big\}_{1\le i\le m}\Big)\,dW(y) \\
&= \int F\big(x_1(\psi),\cdots,x_m(\psi)\big)\prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle\psi_k, C(t_k)^{-1}\psi_k\rangle}\frac{1}{\sqrt{\det[C(t_k)]}}\frac{d^m\psi_k}{\sqrt{(2\pi)^m}} \\
&= \int F(x_{1,T},\cdots,x_{m,T})\prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle x_{t_k}-x_{t_{k-1}},[\Delta t\,C(t_k)]^{-1}(x_{t_k}-x_{t_{k-1}})\rangle}\frac{1}{\sqrt{\det[\Delta t\,C(t_k)]}}\frac{d^m x_{t_k}}{\sqrt{(2\pi)^m}} \\
&= \int F(x_{1,T},\cdots,x_{m,T})\prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle x_{t_k}-x_{t_{k-1}},C_{\Delta t}(t_k)^{-1}(x_{t_k}-x_{t_{k-1}})\rangle}\frac{d^m x_{t_k}}{\sqrt{(2\pi)^m\det[C_{\Delta t}(t_k)]}} \qquad (14.70)
\end{aligned}
$$

where we abbreviated

$$
C_{\Delta t}(t_k) \;:=\; \Delta t\,\times\,C(t_k) \qquad (14.71)
$$

which has matrix entries

$$
c_{i,j}^{\Delta t}(t_k) \;=\; \sigma_{i,t_k}\sigma_{j,t_k}\,\rho_{i,j}(t_k)\,\Delta t \qquad (14.72)
$$

Now, for any symmetric and positive definite matrix $C = (c_{ij})_{1\le i,j\le m} \in \mathbb{R}^{m\times m}$ we introduce the kernel $(x, y \in \mathbb{R}^m)$

$$
p_C(x,y) \;:=\; \frac{1}{\sqrt{(2\pi)^m\det C}}\,e^{-\frac{1}{2}\langle x-y, C^{-1}(x-y)\rangle} \qquad (14.73)
$$

which has the following basic property which is proven in Lemma 14.8 below:

$$\int_{\mathbb{R}^m} p_{C_1}(x,y)p_{C_2}(y,z)\, d^m y \;=\; p_{C_1+C_2}(x,z) \tag{14.74}$$

With this, (14.70) can be rewritten as follows:

$$\begin{aligned}
&\int F\Big( \big\{ \textstyle\int_0^T \sigma_{i,t}\vec{a}_i(t) dy_t \big\}_{1\le i\le m} \Big)\, dW(y) \\
&= \int F(x_{1,T},\cdots,x_{m,T}) \prod_{k=1}^{N_T} e^{-\frac{1}{2}\langle x_{t_k}-x_{t_{k-1}}, C_{\Delta t}(t_k)^{-1}(x_{t_k}-x_{t_{k-1}})\rangle} \frac{d^m x_{t_k}}{\sqrt{(2\pi)^m \det[C_{\Delta t}(t_k)]}} \\
&= \int F(x_{1,T},\cdots,x_{m,T}) \prod_{k=1}^{N_T} p_{C_{\Delta t}(t_k)}(x_{t_{k-1}}, x_{t_k})\, d^m x_{t_k} \\
&\overset{(14.74)}{=} \int_{\mathbb{R}^m} F(x_{1,T},\cdots,x_{m,T})\, p_{\mathrm{sum_}C_{\Delta t}(t_k)}(x_0, x_T)\, d^m x_T
\end{aligned} \tag{14.75}$$

where the matrix $\mathrm{sum\_}C_{\Delta t}(t_k)$ is given by

$$\mathrm{sum\_}C_{\Delta t}(t_k) \;=\; \sum_{k=1}^{N_T} C_{\Delta t}(t_k) \;=:\; C_T \tag{14.76}$$

and has matrix entries

$$\begin{aligned}
c_{ij,T} &= \sum_{k=1}^{N_T} \Delta t \times C(t_k) \\
&= \sum_{k=1}^{N_T} \Delta t \times \sigma_{i,t_k}\sigma_{j,t_k}\, \rho_{i,j}(t_k) \\
&\overset{\Delta t\to 0}{\to} \textstyle\int_0^T \sigma_{i,t}\sigma_{j,t}\, \rho_{i,j}(t)\, dt
\end{aligned} \tag{14.77}$$

Since

$$p_{C_T}(x_0,x_T) \;=\; p_{C_T}(0,x_T) \;=\; \frac{1}{\sqrt{(2\pi)^m \det C_T}}\, e^{-\frac{1}{2}\langle x_T, C_T^{-1} x_T\rangle} \tag{14.78}$$

part (a) follows. To obtain part (b), we apply the formula of part (a) to the expectation value in Theorem 14.5 to obtain

$$V_0 \;=\; e^{-rT} \int_{\mathbb{R}^m} H\Big( \Big\{ S_{i,0}\, e^{x_{i,T} + \int_0^t (r-\sigma_{i,u}^2/2) du} \Big\}_{1\le i\le m} \Big)\; \frac{e^{-\frac{1}{2}\langle x_T, C_T^{-1}x_T\rangle} d^m x_T}{\sqrt{(2\pi)^m \det C_T}} \tag{14.79}$$

Recall that

$$\textstyle\int_0^T \sigma_{i,t}^2 dt \;=\; \sigma_{\mathrm{imp},T}^2\, T \tag{14.80}$$

such that, since $\rho_{ii} = 1$,

$$c_{ii,T} \;=\; \int_0^T \sigma_{i,t}^2 dt \;=\; (\sigma_{i,T}^{\text{imp}})^2\, T \tag{14.81}$$

Thus, with the substitution of variables

$$x_{i,T} \;=\; \sigma_{i,T}^{\text{imp}} \sqrt{T}\, y_i \;=\; \sqrt{c_{ii,T}}\, y_i \tag{14.82}$$

and abbreviating $D = \text{diag}(\sqrt{c_{11,T}}, \cdots, \sqrt{c_{mm,T}}) \in \mathbb{R}^{m\times m}$, equation (14.79) becomes, using $x = Dy$,

$$\begin{aligned} V_0 &= e^{-rT} \int_{\mathbb{R}^m} H\Big( \Big\{ S_{i,0}\, e^{\sigma_{i,T}^{\text{imp}}\sqrt{T}\, y_i + \int_0^t (r-\sigma_{i,u}^2/2)du} \Big\}_{1\le i\le m} \Big)\; e^{-\frac{1}{2}\langle Dy, C_T^{-1} Dy\rangle} \frac{|\det D|\, d^m y}{\sqrt{(2\pi)^m \det C_T}} \\ &= e^{-rT} \int_{\mathbb{R}^m} H\Big( \Big\{ S_{i,0}\, e^{\sigma_{i,T}^{\text{imp}}\sqrt{T}\, y_i + \int_0^t (r-\sigma_{i,u}^2/2)du} \Big\}_{1\le i\le m} \Big)\; \frac{e^{-\frac{1}{2}\langle y, [D^{-1} C_T D^{-1}]^{-1} y\rangle}\; d^m y}{\sqrt{(2\pi)^m \det[D^{-1} C_T D^{-1}]}} \\ &= e^{-rT} \int_{\mathbb{R}^m} H\Big( \Big\{ S_{i,0}\, e^{\sigma_{i,T}^{\text{imp}}\sqrt{T}\, y_i + \int_0^t (r-\sigma_{i,u}^2/2)du} \Big\}_{1\le i\le m} \Big)\; \frac{e^{-\frac{1}{2}\langle y, \rho_{\text{imp}}^{-1} y\rangle}\; d^m y}{\sqrt{(2\pi)^m \det[\rho_{\text{imp}}]}} \end{aligned} \tag{14.83}$$

which proves the theorem. ■.

**Example 14.7:** Consider 2 underlyings with dynamics

$$dS_{1,t}/S_{1,t} \;=\; \mu_1\, dt \;+\; \sigma_1\, dx_{1,t} \tag{14.84}$$

$$dS_{2,t}/S_{2,t} \;=\; \mu_2\, dt \;+\; \sigma_2\, dx_{2,t} \tag{14.85}$$

with constant volatilities and constant correlation $\rho$,

$$dx_{1,t}\, dx_{2,t} \;=\; \rho\, dt \tag{14.86}$$

Then the Black-Scholes fair value $V_0$ of the two-asset option $H$ with payoff (14.2),

$$H(S_{1,T}, S_{2,T}) \;=\; \frac{S_{1,T}/S_{1,0}}{S_{2,T}/S_{2,0}}$$

is given by

$$V_0 \;=\; e^{-rT}\, e^{(\sigma_2^2 - \sigma_1\sigma_2\rho)T} \tag{14.87}$$

In particular, for $\sigma_1 = \sigma_2 = \sigma$,

$$V_0 \;=\; e^{-rT}\, e^{(1-\rho)\sigma^2 T} \tag{14.88}$$

**Proof:** We use the pricing formula (14.58) with Cholesky root

$$\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ \rho\, y_1 + \sqrt{1-\rho^2}\, y_2 \end{pmatrix} \tag{14.89}$$

with uncorrelated $y_1$ and $y_2$. Since

$$\begin{aligned} \frac{S_{1,T}/S_{1,0}}{S_{2,T}/S_{2,0}} &= \frac{e^{\sigma_1\sqrt{T}\,x_1+(r-\sigma_1^2/2)T}}{e^{\sigma_2\sqrt{T}\,x_2+(r-\sigma_2^2/2)T}} \\ &= e^{(\sigma_2^2-\sigma_1^2)\frac{T}{2}}\, e^{\sqrt{T}(\sigma_1 x_1-\sigma_2 x_2)} \\ &\overset{(14.89)}{=} e^{(\sigma_2^2-\sigma_1^2)\frac{T}{2}}\, e^{\sqrt{T}(\sigma_1 y_1-\sigma_2[\rho y_1+\sqrt{1-\rho^2}\,y_2])} \\ &= e^{(\sigma_2^2-\sigma_1^2)\frac{T}{2}}\, e^{\sqrt{T}([\sigma_1-\sigma_2\rho]y_1+\sigma_2\sqrt{1-\rho^2}\,y_2])} \end{aligned}$$

we have to calculate the 2-dimensional integral

$$\begin{aligned} \int_{\mathbb{R}^2} e^{\sqrt{T}([\sigma_1-\sigma_2\rho]y_1+\sigma_2\sqrt{1-\rho^2}\,y_2])}\, e^{-\frac{1}{2}(y_1^2+y_2^2)}\,\frac{dy_1 dy_2}{2\pi} &= e^{[\sigma_1-\sigma_2\rho]^2T/2+\sigma_2^2(1-\rho^2)T/2} \\ &= e^{(\sigma_1^2-2\sigma_1\sigma_2\rho+\sigma_2^2)T/2} \end{aligned} \tag{14.90}$$

such that we end up with

$$\begin{aligned} V_0 &= e^{-rT}\, e^{(\sigma_2^2-\sigma_1^2)\frac{T}{2}}\, e^{(\sigma_1^2-2\sigma_1\sigma_2\rho+\sigma_2^2)T/2} \\ &= e^{-rT}\, e^{(\sigma_2^2-\sigma_1\sigma_2\rho)T} \end{aligned}$$

which is identical to formula (14.88). ■

Finally we have to prove the following lemma which we used in the proof of Theorem 14.6 where we had to calculate an expectation value with respect to Wiener measure with $m$ correlated Brownian motions.

**Lemma 14.8:** Let $C_1$ and $C_2$ be symmetric and positive definite matrices in $\mathbb{R}^{m\times m}$. For $x, y \in \mathbb{R}^m$, define the kernel

$$p_C(x,y) := \frac{1}{\sqrt{(2\pi)^m \det C}}\, e^{-\frac{1}{2}\langle x-y, C^{-1}(x-y)\rangle} \tag{14.91}$$

for a symmetric and positive definite $C$. Then there is the following identity:

$$\int_{\mathbb{R}^m} p_{C_1}(x,y) p_{C_2}(y,z)\, d^m y \quad = \quad p_{C_1+C_2}(x,z) \tag{14.92}$$

**Proof:** ..to be included..