Chapter 16

Girsanov’s Theorem for Ito-Diffusions

The goal in this section is to prove Theorem 16.1 below and provide some application. However, the main use of the Girsanov theorem for us will be that it is a central mathematical tool which allows us to show that exact payoff replication is not just possible in the Black-Scholes model, but is also possible in the presence of stochastic rates (then we have to use also bonds as hedge instruments, not just the stock) and in the presence of stochastic volatility (in that case we also have to use plain vanilla options as hedge instruments for some more complicated path-dependent option which is to be replicated).

More mathematically, we also could say that it is a tool which is used to calculate the equivalent martingale measures for stochastic volatility and stochastic interest rate models.

Let

\[ dW_{0,T} \left( \{ x_t \}_{0 \leq t \leq T} \right) = \lim_{\Delta t \to 0} \frac{N_T}{\Delta t} \prod_{j=1}^{N_T} p_{\Delta t}(x_{t_{j-1}}, x_{t_j}) \, dx_{t_j} \quad (16.1) \]

be the Wiener measure where \( t_j = j \Delta t \), \( N_T = T/\Delta t \) and

\[ p_t(x, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2t}} \quad (16.2) \]

Let

\[ u_t = u(t, \{ x_s \}_{0 \leq s \leq t}) \quad (16.3) \]

be some function which depends on the Brownian motion \( x_s \) up to time \( t \). Make the following substitution of variables \( \{ x_t \}_{0 < t \leq T} \to \{ y_t \}_{0 < t \leq T} \) where

\[ y_t = x_t + \int_0^t u_s \, ds \quad (16.4) \]

In discretized time this reads

\[ y_{t_k} = x_{t_k} + \sum_{j=0}^{k-1} u_{t_j} \Delta t \quad (16.5) \]
or

\[ y_{tk} = x_{tk} + \sum_{j=1}^{k} u_{tj} \Delta t \]  

(16.6)

In the limit \( \Delta t \to 0 \), it does not matter whether we choose (16.5) or (16.6). However, since (16.5) simplifies the computation we will use that choice. In the exercises it is shown that (16.6) leads to the same result, although, for example, the determinant in (16.8) below is not necessarily 1.

Since \( u_t \) depends on \( x_s \) only up to time \( t \), we have

\[ y_{tk} = y_{tk}(\{x_{tj}\}_{0<j\leq k}), \quad x_{tk} = x_{tk}(\{y_{tj}\}_{0<j\leq k}) \]  

(16.7)

and

\[ \det \frac{\partial y}{\partial x} = 1 \]  

(16.8)

Because of

\[ y_{tk} - y_{tk-1} = x_{tk} - x_{tk-1} + u_{tk-1} \Delta t \]  

(16.9)

we have

\[ p\Delta t(y_{tj-1}, y_{tj}) = p\Delta t(x_{tj-1}, x_{tj}) e^{-\frac{1}{2} \Delta t} \]  

(16.10)

which gives

\[ dW_{0,T}(\{y_t\}) = dW_{0,T}(\{x_t\}) e^{-\frac{1}{2} \Delta t} \]  

(16.11)

Suppose now that \( X_t \) is an Ito diffusion given by the equation

\[ dX_t = \mu_t dt + \sigma_t dx_t \]  

(16.12)

where \( x_t \) is a Brownian motion and \( \mu_t = \mu(t, X_t) \), the same for \( \sigma \). One may also consider the case \( \mu_t = \mu(t, \{x_s\}_{0<s\leq t}) \). The discretized version of (16.12) reads

\[ X_{tk} - X_{tk-1} = \mu_{tk-1} \Delta t + \sigma_{tk-1}(x_{tk} - x_{tk-1}) \]  

(16.13)

which determines \( X_{tk} \) as a function of the \( \{x_{tj}\}_{0<j\leq k} \),

\[ X_t = X_t(\{x_s\}_{0<s\leq t}) \]  

(16.14)

Now let

\[ r_t = r(t, \{x_s\}_{0<s\leq t}) \]  

(16.15)
be some function and define
\[ u_t := \frac{\mu_t - r_t}{\sigma_t} \]  
which again depends on the Brownian motion \( x_s \) only up to time \( t \). Make the substitution of variables (16.4),
\[ y_t = x_t + \int_0^t u_s \, ds \]
Let \( X_t \), considered as a function of the new \( y_s \) variables, be denoted as \( Y_t \). That is,
\[ Y_t(\{y_s\}_{0<s\leq t}) := X_t(\{x_s\}_{0<s\leq t}) \quad \text{where} \quad x_s = x_s(\{y_u\}_{0<u\leq s}) \]  
What is the SDE for \( Y_t \), in terms of the new variables \( y_s \)? Since
\[ dy_t = dx_t + u_t \, dt \]
we have
\[ dY_t(y) = dX_t(x) \]
\[ = \mu_t \, dt + \sigma_t \, dx_t \]
\[ = \mu_t \, dt + \sigma_t \left( dy_t - \frac{\mu_t - r_t}{\sigma_t} \, dt \right) \]
\[ = r_t \, dt + \sigma_t \, dy_t \]
Thus we have proven the following

**Theorem 16.1:** a) Let \( dW_{[0,T]}(\{x_t\}_{0<t\leq T}) \) be the Wiener measure and let \( u_t = u(t, \{x_s\}_{0<s\leq t}) \) be some function. Make the substitution of variables \( \{x_t\}_{0<t\leq T} \rightarrow \{y_t\}_{0<t\leq T} \) where
\[ y_t = x_t + \int_0^t u_s \, ds \]  
Then
\[ dW_{[0,T]}(\{y_t\}_{0<t\leq T}) = e^{-\int_0^T u_s \, ds - \frac{1}{2} \int_0^T \sigma_s^2 \, ds} \, dW_{[0,T]}(\{x_t\}_{0<t\leq T}) \]

b) Let \( X_t \) be the solution of the SDE
\[ dX_t = \mu_t \, dt + \sigma_t \, dx_t, \quad X_0 = a \]
where \( x_t \) is a Brownian motion and \( \mu_t = \mu(t, X_t) \), the same for \( \sigma_t \), or more general \( \mu_t = \mu(t, \{x_s\}_{0<s\leq t}) \), the same for \( \sigma_t \). Let \( r_t = r(t, \{x_s\}_{0<s\leq t}) \) be some function and define \( u_t := (\mu_t - r_t)/\sigma_t \). Make the substitution of variables (16.20) and define
\[ Y_t(\{y_s\}_{0<s\leq t}) := X_t(\{x_s\}_{0<s\leq t}) \quad \text{where} \quad x_s = x_s(\{y_u\}_{0<u\leq s}) , \]
\( x = x(y) \) given by (16.20). Then \( Y_t \) is a solution of the SDE
\[
dY_t = r_t \, dt + \sigma_t \, dy_t, \quad Y_0 = a
\] (16.24)
and there is the following equality of expectations:
\[
\int F(\{Y_t(y)\}_{0 \leq t \leq T}) \, dW(0,T](y) = \int F(\{X_t(x)\}_{0 \leq t \leq T}) \, e^{-\int_0^T u_s \, dx_s - \frac{1}{2} \int_0^T u_s^2 \, ds} \, dW(0,T](x)
\] (16.25)
where \( F \) is some function.

**Applications**

In Chapter 20 we discuss Bessel processes. A squared Bessel process \( X_t \) is a solution of the SDE
\[
dX_t = n \, dt + 2 \sqrt{X_t} \, dx_t
\] (16.26)
It can be realized as the sum of squares of \( n \) independent Brownian motions. A Cox-Ingersoll-Ross process \( Y_t \) is the solution of the SDE
\[
dY_t = (a - b Y_t) \, dt + \sigma \sqrt{Y_t} \, dy_t
\] (16.27)
where \( y_t \) and \( x_t \) are Brownian motions. Suppose we know the expectation \( E[e^{\lambda X_T} - c \int_0^T X_s \, ds} \) (the expectation meaning integration with respect to \( dW(0,T](x) \)), can we derive from this the value of \( E[e^{\lambda Y_T} - c \int_0^T Y_s \, ds} \) (the expectation now meaning integration with respect to \( dW(0,T](y) \))? In order to apply (16.25), the coefficients in front of the Brownian motions in (16.26) and (16.27) have to coincide. To this end define the process \( U_t := \frac{\sigma^2}{4} X_t \). Then
\[
E[e^{\frac{\lambda^2}{2} Y_T - \frac{\sigma^2}{8} \int_0^T Y_s \, ds}] = E[e^{\frac{\lambda^2}{2} U_T - \frac{\sigma^2}{8} \int_0^T U_s \, ds}]
\]
and \( U_t \) satisfies the SDE
\[
dU_t = a \, dt + \sigma \sqrt{U_t} \, dx_t
\] (16.28)
where we put \( a = \frac{\sigma^2 n}{4} \). In order to obtain (16.27) from (16.28), we have to define
\[
u_t = \frac{a - (a - b Y_t)}{\sigma \sqrt{Y_t}} = \frac{b}{\sigma} \sqrt{Y_t}
\] (16.29)
and make the substitution of variables (16.20). Then we can use (16.25) to obtain (recall that \( Y(y) = U(x) \))
\[
E[e^{\frac{\lambda^2}{2} Y_T - \frac{\sigma^2}{8} \int_0^T Y_s \, ds}] = \int e^{\frac{\lambda^2}{2} Y_T - \frac{\sigma^2}{8} \int_0^T Y_s \, ds} \, dW(0,T](y)
\]
\[
= \int e^{\frac{\lambda^2}{2} U_T - \frac{\sigma^2}{8} \int_0^T U_s \, ds} e^{-\int_0^T u_s \, dx_s - \frac{1}{2} \int_0^T u_s^2 \, ds} \, dW(0,T](x)
\] (16.30)
Because of

\[ \int_0^T u_s \, dx_s = \frac{b}{\sigma} \int_0^T \sqrt{Y_s(y)} \, dx_s = \frac{b}{\sigma^2} (U_T - U_0 - aT) \quad (16.29) \]

and

\[ \int_0^T u_s^2 \, ds = \frac{b^2}{\sigma^2} \int_0^T Y_s(y) \, ds = \frac{b^2}{\sigma^2} \int_0^T U_s(x) \, ds \quad (16.28) \]

we arrive at

\[ \mathbb{E}\left[e^{\frac{1}{2} Y_T - \frac{b^2}{2\sigma^2} \int_0^T Y_s ds}\right] = \int e^{\frac{1}{2} Y_T - \frac{1}{2} \int_0^T U_s ds} e^{-\frac{b}{\sigma}(U_T-U_0-aT)-\frac{b^2}{2\sigma^2} \int_0^T U_s(x)ds} dW_{(0,T]}(x) \]

\[ = e^{-\frac{b}{\sigma^2} (U_0-aT)} \int e^{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right) U_T - \left(\frac{1}{2} \frac{b^2}{\sigma^2}+\frac{b^2}{\sigma}\right) \int_0^T U_s ds} dW_{(0,T]}(x) \]

\[ = e^{\frac{b}{\sigma^2} (Y_0+aT)} \int e^{\left(\frac{1}{2} \frac{b^2}{\sigma^2} - \frac{b}{\sigma}\right) X_T - \left(\frac{b^2}{\sigma^2} + \frac{b^2}{\sigma}\right) \int_0^T X_s ds} dW_{(0,T]}(x) \quad (16.33) \]

The right hand side of (16.33) we compute in chapter 21 using martingale methods. In later chapters we need actually the higher dimensional version of Theorem 16.1 which reads as follows:

**Theorem 16.2:** a) Let \( \vec{x}_t = (x_1^t, ..., x_n^t) \) be an uncorrelated n-dimensional Brownian motion and let

\[ dW_{(0,T]}\left(\{\vec{x}_t\}_{0 < t \leq T}\right) = \lim_{\Delta t \to 0} \prod_{j=1}^{N_T} p_{\Delta t}(\vec{x}_{t_{j-1}}, \vec{x}_{t_j}) \, dp_{\Delta t} \quad (16.34) \]

be the Wiener measure where \( t_j = j \Delta t, N_T = T/\Delta t \) and

\[ p_t(\vec{x}, \vec{y}) = \frac{1}{\sqrt{2\pi^t}} e^{-\frac{\langle \vec{x} - \vec{y} \rangle^2}{2t}} \quad (16.35) \]

Let \( \vec{u}_t = \vec{u}(t, \{\vec{x}_s\}_{0 < s \leq t}) \in \mathbb{R}^n \) be some function. Make the substitution of variables \( \{\vec{x}_t\}_{0 < t \leq T} \to \{\vec{y}_t\}_{0 < t \leq T} \) where

\[ \vec{y}_t = \vec{x}_t + \int_0^t \vec{u}_s \, ds \quad (16.36) \]

Then

\[ dW_{(0,T]}\left(\{\vec{y}_t\}_{0 < t \leq T}\right) = e^{-\frac{1}{2} \int_0^T \vec{u}_s^2 \, ds - \frac{1}{2} \int_0^T \vec{u}_s^2 \, ds} dW_{(0,T]}\left(\{\vec{x}_t\}_{0 < t \leq T}\right) \quad (16.37) \]
b) Let \( X_t \in \mathbb{R}^m \) be the solution of the SDE

\[
\begin{align*}
\frac{dX_t}{dt} &= \bar{\mu}_t dt + \sigma_t d\bar{x}_t,
X_0 = \bar{a}
\end{align*}
\]  

(16.38)

where \( \bar{x}_t \) is an \( n \)-dimensional Brownian motion and \( \bar{\mu}_t = \bar{\mu}(t, X_t) \in \mathbb{R}^m \), or more general \( \bar{\mu}_t = \bar{\mu}(t, \{\bar{x}_s\}_{0 < s \leq t}) \), and \( \sigma_t = \sigma(t, X_t) \in \mathbb{R}^{m \times n} \) or more general \( \sigma_t = \sigma(t, \{\bar{x}_s\}_{0 < s \leq t}) \). Let \( \bar{r}_t = \bar{r}(t, \{\bar{x}_s\}_{0 < s \leq t}) \in \mathbb{R}^m \) and \( \bar{u}_t = \bar{u}(t, \{\bar{x}_s\}_{0 < s \leq t}) \in \mathbb{R}^n \) be some functions such that

\[
\sigma_t \bar{u}_t = \bar{\mu}_t - \bar{r}_t
\]  

(16.39)

Make the substitution of variables (16.36) and define

\[
Y_t(\{\bar{y}_s\}_{0 < s \leq t}) := X_t(\{\bar{x}_s\}_{0 < s \leq t}) \quad \text{where} \quad \bar{x}_s = \bar{x}(\{\bar{y}_u\}_{0 < u \leq s}),
\]  

(16.40)

\( \bar{x} = \bar{x}(\bar{y}) \) given by (16.36). Then \( Y_t \) is a solution of the SDE

\[
\frac{dY_t}{dt} = \bar{r}_t dt + \sigma_t d\bar{y}_t, \quad Y_0 = \bar{a}
\]  

(16.41)

and there is the following equality of expectations:

\[
\int F(\{Y_t(\bar{y})\}_{0 \leq t \leq T}) dW(0, T)(\bar{y}) = \int F(\{X_t(\bar{x})\}_{0 \leq t \leq T}) e^{-\int_0^T \bar{u}_s d\bar{x}_s - \frac{1}{2} \int_0^T \bar{u}_s^2 ds} dW(0, T)(\bar{x})
\]  

(16.42)

where \( F \) is some function.

**Proof:** The proof is analog to the one-dimensional case. Observe that, if we make the substitution of variables

\[
\bar{y}_{k} = \bar{x}_k + \sum_{j=0}^{k-1} \bar{u}_j \Delta t
\]  

(16.43)

then, since \( \bar{u}_j = \bar{u}(t_j, \{\bar{x}_t\}_{0 < t \leq j}) \), the Jacobian

\[
\det \frac{\partial \bar{y}}{\partial \bar{x}} = \det \begin{pmatrix}
\frac{\partial \bar{y}_1}{\partial \bar{x}_1} & \frac{\partial \bar{y}_1}{\partial \bar{x}_2} & \cdots & \frac{\partial \bar{y}_1}{\partial \bar{x}_N} \\
\frac{\partial \bar{y}_2}{\partial \bar{x}_1} & \frac{\partial \bar{y}_2}{\partial \bar{x}_2} & \cdots & \frac{\partial \bar{y}_2}{\partial \bar{x}_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \bar{y}_N}{\partial \bar{x}_1} & \frac{\partial \bar{y}_N}{\partial \bar{x}_2} & \cdots & \frac{\partial \bar{y}_N}{\partial \bar{x}_N}
\end{pmatrix} = \det \begin{pmatrix}
\text{Id}_n & & \\
0 & \text{Id}_n & & \\
& \ddots & \ddots & \\
& & \ast & \text{Id}_n
\end{pmatrix} = 1
\]  

(16.44)

since the matrix is still lower triangular with entries 1 on the diagonal. \( \text{Id}_n \) denotes the \( n \times n \) identity matrix and \( \ast \) stands for some arbitrary \( n \times n \) matrix. Then (16.37) follows
in the same way as (16.11). The analog equation of (16.45) reads

\[
dY_t(\vec{y}) = dX_t(\vec{x}) = \vec{\mu}_t \, dt + \sigma_t \, d\vec{x}_t \tag{16.36}
\]

\[
= \vec{\mu}_t \, dt + \sigma_t \left( d\vec{y}_t - \vec{\mu}_t \, dt \right) \tag{16.39}
\]

\[
= \vec{r}_t \, dt + \sigma_t \, d\vec{y}_t
\]

This proves the theorem. ■