Chapter 16

Girsanov's Theorem for Ito-Diffusions

The goal in this section is to prove Theorem 16.1 below and provide some application. However, the main use of the Girsanov theorem for us will be that it is a central mathematical tool which allows us to show that exact payoff replication is not just possible in the Black-Scholes model, but is also possible in the presence of stochastic rates (then we have to use also bonds as hedge instruments, not just the stock) and in the presence of stochastic volatility (in that case we also have to use plain vanilla options as hedge instruments for some more complicated path-dependent option which is to be replicated).

More mathematically, we also could say that it is a tool which is used to calculate the equivalent martingale measures for stochastic volatility and stochastic interest rate models.

Let

$$dW_{(0,T]}\big(\{x_t\}_{0<t\leq T}\big) \;=\; \lim_{\Delta t\to 0} \prod_{j=1}^{N_T} p_{\Delta t}(x_{t_{j-1}}, x_{t_j})\, dx_{t_j} \tag{16.1}$$

be the Wiener measure where $t_j = j\Delta t$, $N_T = T/\Delta t$ and

$$p_t(x,y) \;=\; \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-y)^2}{2t}} \tag{16.2}$$

Let

$$u_t \;=\; u\big(t, \{x_s\}_{0<s\leq t}\big) \tag{16.3}$$

be some function which depends on the Brownian motion x_s up to time t. Make the following substitution of variables $\{x_t\}_{0<t\leq T} \to \{y_t\}_{0<t\leq T}$ where

$$y_t \;=\; x_t + \int_0^t u_s\, ds \tag{16.4}$$

In discretized time this reads

$$y_{t_k} \;=\; x_{t_k} + \sum_{j=0}^{k-1} u_{t_j}\, \Delta t \tag{16.5}$$

or

$$y_{t_k} \;=\; x_{t_k} + \sum_{j=1}^{k} u_{t_j}\,\Delta t \tag{16.6}$$

In the limit $\Delta t \to 0$, it does not matter whether we choose (16.5) or (16.6). However, since (16.5) simplifies the computation we will use that choice. In the exercises it is shown that (16.6) leads to the same result, although, for example, the determinant in (16.8) below is not necessarily 1.

Since u_t depends on x_s only up to time t, we have

$$y_{t_k} = y_{t_k}\big(\{x_{t_j}\}_{0<j\le k}\big)\,, \qquad x_{t_k} = x_{t_k}\big(\{y_{t_j}\}_{0<j\le k}\big) \tag{16.7}$$

and

$$\det \tfrac{\partial y}{\partial x} = 1 \tag{16.8}$$

Because of

$$y_{t_k} - y_{t_{k-1}} \;=\; x_{t_k} - x_{t_{k-1}} + u_{t_{k-1}}\Delta t \tag{16.9}$$

we have

$$p_{\Delta t}(y_{t_{j-1}}, y_{t_j}) \;=\; p_{\Delta t}(x_{t_{j-1}}, x_{t_j})\, e^{-(x_{t_j}-x_{t_{j-1}})u_{t_{j-1}}}\, e^{-\frac{u_{t_{j-1}}^2}{2}\Delta t} \tag{16.10}$$

which gives

$$dW_{(0,T]}\big(\{y_t\}\big) \;=\; dW_{(0,T]}\big(\{x_t\}\big)\, e^{-\int_0^T u_s dx_s - \frac{1}{2}\int_0^T u_s^2 ds} \tag{16.11}$$

Suppose now that X_t is an Ito diffusion given by the equation

$$dX_t \;=\; \mu_t\, dt + \sigma_t\, dx_t \tag{16.12}$$

where x_t is a Brownian motion and $\mu_t = \mu(t, X_t)$, the same for σ. One may also consider the case $\mu_t = \mu\big(t, \{x_s\}_{0<s\le t}\big)$. The discretized version of (16.12) reads

$$X_{t_k} - X_{t_{k-1}} \;=\; \mu_{t_{k-1}}\Delta t + \sigma_{t_{k-1}}(x_{t_k} - x_{t_{k-1}}) \tag{16.13}$$

which determines X_{t_k} as a function of the $\{x_{t_j}\}_{0<j\le k}$,

$$X_t = X_t\big(\{x_s\}_{0<s\le t}\big) \tag{16.14}$$

Now let

$$r_t \;=\; r\big(t, \{x_s\}_{0<s\le t}\big) \tag{16.15}$$

be some function and define

$$u_t \quad := \quad \frac{\mu_t - r_t}{\sigma_t} \tag{16.16}$$

which again depends on the Brownian motion x_s only up to time t. Make the substitution of variables (16.4),

$$y_t \quad = \quad x_t + \int_0^t u_s \, ds$$

Let X_t, considered as a function of the new y_s variables, be denoted as Y_t. That is,

$$Y_t\big(\{y_s\}_{0<s\le t}\big) \quad := \quad X_t\big(\{x_s\}_{0<s\le t}\big) \quad \text{where} \;\; x_s \;=\; x_s\big(\{y_u\}_{0<u\le s}\big) \tag{16.17}$$

What is the SDE for Y_t, in terms of the new variables y_s? Since

$$\begin{aligned} dy_t &= dx_t + u_t \, dt \\ &= dx_t + \tfrac{\mu_t - r_t}{\sigma_t} \, dt \end{aligned} \tag{16.18}$$

we have

$$\begin{aligned} dY_t(y) &= dX_t(x) \\ &= \mu_t \, dt + \sigma_t \, dx_t \\ &\overset{(16.18)}{=} \mu_t \, dt + \sigma_t \left(dy_t - \tfrac{\mu_t - r_t}{\sigma_t} \, dt \right) \\ &= r_t \, dt + \sigma_t \, dy_t \end{aligned} \tag{16.19}$$

Thus we have proven the following

Theorem 16.1: a) Let $dW_{(0,T]}\big(\{x_t\}_{0<t\le T}\big)$ be the Wiener measure and let $u_t = u\big(t, \{x_s\}_{0<s\le t}\big)$ be some function. Make the substitution of variables $\{x_t\}_{0<t\le T} \to \{y_t\}_{0<t\le T}$ where

$$y_t \quad = \quad x_t + \int_0^t u_s \, ds \tag{16.20}$$

Then

$$dW_{(0,T]}\big(\{y_t\}_{0<t\le T}\big) \quad = \quad e^{-\int_0^T u_s dx_s - \frac{1}{2}\int_0^T u_s^2 ds} \; dW_{(0,T]}\big(\{x_t\}_{0<t\le T}\big) \tag{16.21}$$

b) Let X_t be the solution of the SDE

$$dX_t \quad = \quad \mu_t \, dt + \sigma_t \, dx_t \, , \qquad X_0 = a \tag{16.22}$$

where x_t is a Brownian motion and $\mu_t = \mu(t, X_t)$, the same for σ_t, or more general $\mu_t = \mu\big(t, \{x_s\}_{0<s\le t}\big)$, the same for σ_t. Let $r_t = r\big(t, \{x_s\}_{0<s\le t}\big)$ be some function and define $u_t := (\mu_t - r_t)/\sigma_t$. Make the substitution of variables (16.20) and define

$$Y_t\big(\{y_s\}_{0<s\le t}\big) \quad := \quad X_t\big(\{x_s\}_{0<s\le t}\big) \quad \text{where} \;\; x_s \;=\; x_s\big(\{y_u\}_{0<u\le s}\big) \, , \tag{16.23}$$

$x = x(y)$ given by (16.20). Then Y_t is a solution of the SDE

$$dY_t \;=\; r_t\, dt + \sigma_t\, dy_t\,, \quad Y_0 = a \tag{16.24}$$

and there is the following equality of expectations:

$$\int F\big(\{Y_t(y)\}_{0\le t\le T}\big)\, dW_{(0,T]}(y) \;=\; \int F\big(\{X_t(x)\}_{0\le t\le T}\big)\, e^{-\int_0^T u_s dx_s - \frac{1}{2}\int_0^T u_s^2 ds}\, dW_{(0,T]}(x) \tag{16.25}$$

where F is some function.

Applications

In Chapter 20 we discuss Bessel processes. A squared Bessel process X_t is a solution of the SDE

$$dX_t \;=\; n\, dt + 2\sqrt{X_t}\, dx_t \tag{16.26}$$

It can be realized as the sum of squares of n independent Brownian motions. A Cox-Ingersoll-Ross process Y_t is the solution of the SDE

$$dY_t \;=\; (a - b\, Y_t)\, dt + \sigma\sqrt{Y_t}\, dy_t \tag{16.27}$$

where y_t and x_t are Brownian motions. Suppose we know the expectation $\mathsf{E}\big[e^{\frac{\lambda}{2}X_T - \frac{c}{2}\int_0^T X_s ds}\big]$ (the expectation meaning integration with respect to $dW_{(0,T]}(x)$), can we derive from this the value of $\mathsf{E}\big[e^{\frac{\lambda}{2}Y_T - \frac{c}{2}\int_0^T Y_s ds}\big]$ (the expectation now meaning integration with respect to $dW_{(0,T]}(y)$)? In order to apply (16.25), the coefficients in front of the Brownian motions in (16.26) and (16.27) have to coincide. To this end define the process $U_t := \frac{\sigma^2}{4}X_t$. Then $\mathsf{E}\big[e^{\frac{\lambda}{2}X_T - \frac{c}{2}\int_0^T X_s ds}\big] = \mathsf{E}\big[e^{\frac{2\lambda}{\sigma^2}U_T - \frac{2c}{\sigma^2}\int_0^T U_s ds}\big]$ and U_t satisfies the SDE

$$dU_t \;=\; a\, dt + \sigma\sqrt{U_t}\, dx_t \tag{16.28}$$

where we put $a = \frac{\sigma^2 n}{4}$. In order to obtain (16.27) from (16.28), we have to define

$$u_t \;=\; \frac{a - (a - bY_t)}{\sigma\sqrt{Y_t}} \;=\; \frac{b}{\sigma}\sqrt{Y_t} \tag{16.29}$$

and make the substitution of variables (16.20). Then we can use (16.25) to obtain (recall that $Y(y) = U(x)$)

$$\begin{aligned}\mathsf{E}\Big[e^{\frac{\lambda}{2}Y_T - \frac{c}{2}\int_0^T Y_s ds}\Big] \;&=\; \int e^{\frac{\lambda}{2}Y_T - \frac{c}{2}\int_0^T Y_s ds}\, dW_{(0,T]}(y) \\ &=\; \int e^{\frac{\lambda}{2}U_T - \frac{c}{2}\int_0^T U_s ds}\, e^{-\int_0^T u_s dx_s - \frac{1}{2}\int_0^T u_s^2 ds}\, dW_{(0,T]}(x)\end{aligned} \tag{16.30}$$

Because of

$$\begin{aligned}\int_0^T u_s dx_s &\overset{(16.29)}{=} \tfrac{b}{\sigma}\int_0^T \sqrt{Y_s(y)}\,dx_s \\ &= \tfrac{b}{\sigma}\int_0^T \sqrt{U_s(x)}\,dx_s \\ &\overset{(16.28)}{=} \tfrac{b}{\sigma^2}(U_T - U_0 - aT) \qquad (16.31)\end{aligned}$$

and

$$\begin{aligned}\int_0^T u_s^2 ds &= \tfrac{b^2}{\sigma^2}\int_0^T Y_s(y)ds \\ &= \tfrac{b^2}{\sigma^2}\int_0^T U_s(x)ds \qquad (16.32)\end{aligned}$$

we arrive at

$$\begin{aligned}\mathsf{E}\Big[e^{\frac{\lambda}{2}Y_T - \frac{c}{2}\int_0^T Y_s ds}\Big] &= \int e^{\frac{\lambda}{2}U_T - \frac{c}{2}\int_0^T U_s ds}\, e^{-\frac{b}{\sigma^2}(U_T - U_0 - aT) - \frac{b^2}{2\sigma^2}\int_0^T U_s(x)ds}\, dW_{(0,T]}(x) \\ &= e^{-\frac{b}{\sigma^2}(-U_0 - aT)}\int e^{(\frac{\lambda}{2}-\frac{b}{\sigma^2})U_T - (\frac{c}{2}+\frac{b^2}{2\sigma^2})\int_0^T U_s ds}\, dW_{(0,T]}(x) \\ &= e^{\frac{b}{\sigma^2}(Y_0 + aT)}\int e^{(\frac{\lambda\sigma^2}{8}-\frac{b}{4})X_T - (\frac{c\sigma^2}{8}+\frac{b^2}{8})\int_0^T X_s ds}\, dW_{(0,T]}(x) \qquad (16.33)\end{aligned}$$

The right hand side of (16.33) we compute in chapter 21 using martingale methods. In later chapters we need actually the higher dimensional version of Theorem 16.1 which reads as follows:

Theorem 16.2: a) Let $\vec{x}_t = (x_t^1, ..., x_t^n)$ be an uncorrelated n-dimensional Brownian motion and let

$$dW_{(0,T]}\big(\{\vec{x}_t\}_{0<t\leq T}\big) = \lim_{\Delta t\to 0}\prod_{j=1}^{N_T} p_{\Delta t}(\vec{x}_{t_{j-1}}, \vec{x}_{t_j})\, d^n x_{t_j} \qquad (16.34)$$

be the Wiener measure where $t_j = j\Delta t$, $N_T = T/\Delta t$ and

$$p_t(\vec{x}, \vec{y}) = \tfrac{1}{\sqrt{2\pi}^n}\, e^{-\frac{(\vec{x}-\vec{y})^2}{2t}} \qquad (16.35)$$

Let $\vec{u}_t = \vec{u}\big(t, \{\vec{x}_s\}_{0<s\leq t}\big) \in \mathbb{R}^n$ be some function. Make the substitution of variables $\{\vec{x}_t\}_{0<t\leq T} \to \{\vec{y}_t\}_{0<t\leq T}$ where

$$\vec{y}_t = \vec{x}_t + \int_0^t \vec{u}_s\, ds \qquad (16.36)$$

Then

$$dW_{(0,T]}\big(\{\vec{y}_t\}_{0<t\leq T}\big) = e^{-\int_0^T \vec{u}_s d\vec{x}_s - \frac{1}{2}\int_0^T \vec{u}_s^2 ds}\, dW_{(0,T]}\big(\{\vec{x}_t\}_{0<t\leq T}\big) \qquad (16.37)$$

b) Let $X_t \in \mathbb{R}^m$ be the solution of the SDE

$$dX_t \;=\; \vec{\mu}_t\, dt + \sigma_t\, d\vec{x}_t\,, \qquad X_0 = \vec{a} \tag{16.38}$$

where $\vec{x}_t$ is an n-dimensional Brownian motion and $\vec{\mu}_t = \vec{\mu}(t, X_t) \in \mathbb{R}^m$, or more general $\vec{\mu}_t = \vec{\mu}\big(t, \{\vec{x}_s\}_{0<s\leq t}\big)$, and $\sigma_t = \sigma(t, X_t) \in \mathbb{R}^{m\times n}$ or more general $\sigma_t = \sigma\big(t, \{\vec{x}_s\}_{0<s\leq t}\big)$. Let $\vec{r}_t = \vec{r}\big(t, \{\vec{x}_s\}_{0<s\leq t}\big) \in \mathbb{R}^m$ and $\vec{u}_t = \vec{u}\big(t, \{\vec{x}_s\}_{0<s\leq t}\big) \in \mathbb{R}^n$ be some functions such that

$$\sigma_t\, \vec{u}_t \;=\; \vec{\mu}_t - \vec{r}_t \tag{16.39}$$

Make the substitution of variables (16.36) and define

$$Y_t\big(\{\vec{y}_s\}_{0<s\leq t}\big) \;:=\; X_t\big(\{\vec{x}_s\}_{0<s\leq t}\big) \quad \text{where} \;\; \vec{x}_s \;=\; \vec{x}_s\big(\{\vec{y}_u\}_{0<u\leq s}\big)\,, \tag{16.40}$$

$\vec{x} = \vec{x}(\vec{y})$ given by (16.36). Then Y_t is a solution of the SDE

$$dY_t \;=\; \vec{r}_t\, dt + \sigma_t\, d\vec{y}_t\,, \qquad Y_0 = \vec{a} \tag{16.41}$$

and there is the following equality of expectations:

$$\int F\big(\{Y_t(\vec{y})\}_{0\leq t\leq T}\big)\, dW_{(0,T]}(\vec{y}) \;=\; \int F\big(\{X_t(\vec{x})\}_{0\leq t\leq T}\big)\, e^{-\int_0^T \vec{u}_s d\vec{x}_s - \frac{1}{2}\int_0^T \vec{u}_s^2 ds}\, dW_{(0,T]}(\vec{x}) \tag{16.42}$$

where F is some function.

Proof: The proof is analog to the one-dimensional case. Observe that, if we make the substitution of variables

$$\vec{y}_{t_k} \;=\; \vec{x}_{t_k} + \sum_{j=0}^{k-1} \vec{u}_{t_j}\, \Delta t \tag{16.43}$$

then, since $\vec{u}_{t_j} = \vec{u}\big(t_j, \{\vec{x}_{t_i}\}_{0<i\leq j}\big)$, the Jacobian

$$\det \frac{\partial \vec{y}}{\partial \vec{x}} \;=\; \det \begin{pmatrix} \frac{\partial \vec{y}_{t_1}}{\partial \vec{x}_{t_1}} & \frac{\partial \vec{y}_{t_1}}{\partial \vec{x}_{t_2}} & \cdots & \frac{\partial \vec{y}_{t_1}}{\partial \vec{x}_{t_N}} \\ \frac{\partial \vec{y}_{t_2}}{\partial \vec{x}_{t_1}} & \frac{\partial \vec{y}_{t_2}}{\partial \vec{x}_{t_2}} & \cdots & \frac{\partial \vec{y}_{t_2}}{\partial \vec{x}_{t_N}} \\ \vdots & & & \vdots \\ \frac{\partial \vec{y}_{t_N}}{\partial \vec{x}_{t_1}} & \frac{\partial \vec{y}_{t_N}}{\partial \vec{x}_{t_2}} & \cdots & \frac{\partial \vec{y}_{t_N}}{\partial \vec{x}_{t_N}} \end{pmatrix} \;=\; \det \begin{pmatrix} Id_n & 0 & \cdots & 0 \\ * & Id_n & \cdots & 0 \\ \vdots & & & \vdots \\ * & * & \cdots & Id_n \end{pmatrix} \;=\; 1 \tag{16.44}$$

since the matrix is still lower triangular with entries 1 on the diagonal. Id_n denotes the $n \times n$ identity matrix and $*$ stands for some arbitrary $n \times n$ matrix. Then (16.37) follows in the same way as (16.11). The analog equation of (16.45) reads

$$\begin{aligned} dY_t(\vec{y}) &= dX_t(\vec{x}) \\ &= \vec{\mu}_t\, dt + \sigma_t\, d\vec{x}_t \\ &\overset{(16.36)}{=} \vec{\mu}_t\, dt + \sigma_t\, (d\vec{y}_t - \vec{u}_t\, dt) \\ &\overset{(16.39)}{=} \vec{\mu}_t\, dt + \sigma_t d\vec{y}_t - (\vec{\mu}_t - \vec{r}_t)\, dt \\ &= \vec{r}_t\, dt + \sigma_t\, d\vec{y}_t \qquad (16.45) \end{aligned}$$

This proves the theorem. ■