Chapter 21

The Cox-Ingersoll-Ross Process

In the following we list some named models for the short rate process. We follow the presentation of [4].

The Vasicek Model

The Vasicek model is a mean reverting Ornstein-Uhlenbeck process. It is given by the SDE

$$dr_t \;=\; a(b - r_t)dt + \sigma dB_t \tag{21.1}$$

with non-negative constants a, b and σ and initial value $r_0 > 0$. Its solution is given by (compare lecture 14)

$$r_t \;=\; r_0\, e^{-at} + b(1 - e^{-at}) + \sigma\, e^{-at} \int_0^t e^{as} dB_s \tag{21.2}$$

Thus r_t has a Gaussian distribution function with mean $r_0\, e^{-at} + b(1 - e^{-at})$ and variance $\sigma^2\, e^{-2at} \int_0^t e^{2as} ds = \frac{\sigma^2}{2a}(1 - e^{-2at})$. In particular, the probability for negative values of r_t is non zero which makes the model not too realistic. Its consideration is merely due to its simplicity.

The Hull-White Model

The Hull-White model in its simplest form is a Vasicek model with time dependent, deterministic coefficients. It is given by the SDE

$$dr_t \;=\; \big(\alpha(t) - \beta(t) r_t\big) dt + \sigma(t)\, dB_t \tag{21.3}$$

with some initial value $r_0 > 0$. The solution is ($b(t) = \int_0^t \beta(s)\, ds$)

$$r_t \;=\; e^{-b(t)} \Big(r_0 + \int_0^t e^{b(s)} \alpha(s) ds + \int_0^t e^{b(s)} \sigma(s)\, dB_s \Big) \tag{21.4}$$

which is again a Gaussian process. The model is popular with practitioners. In its more general form it includes a term r_t^γ in the volatility, in which case it generalizes the Cox-Ingersoll-Ross model discussed next.

The Cox-Ingersoll-Ross Model

The following material is taken from section 9.6 of [4]. In the Vasicek and Hull-White model the probability for a negative rate r_t is non zero. The Cox-Ingersoll-Ross model provides an SDE for r_t whose solution is always positve. To motivate it, consider n Ornstein-Uhlenbeck processes $X_i(t)$ given by the SDE's

$$dX_i(t) = -\tfrac{\alpha}{2} X_i(t)\,dt + \tfrac{\sigma}{2} dB_i(t) \tag{21.5}$$

with n uncorrelated Brownian motions $B_1, ..., B_n$. Consider the process

$$r(t) \;:=\; X_1^2(t) + \cdots + X_n^2(t) \tag{21.6}$$

then

$$\begin{aligned} dr(t) &= \sum_{i=1}^n 2X_i(t)\left(-\frac{\alpha}{2}X_i(t)\,dt + \frac{\sigma}{2}dB_i(t)\right) + \tfrac{1}{2}\sum_{i=1}^n 2\frac{\sigma^2}{4}dt \\ &= -\alpha r(t)dt + \sigma\sum_{i=1}^n X_i(t)dB_i(t) + \tfrac{n\sigma^2}{4}dt \\ &= \left(\tfrac{n\sigma^2}{4} - \alpha r(t)\right)dt + \sigma\sqrt{r(t)}\sum_{i=1}^n \tfrac{X_i(t)dB_i(t)}{\sqrt{r(t)}} \end{aligned} \tag{21.7}$$

The process $W(t) := \sum_{i=1}^n \int_0^t \frac{X_i(s)dB_i(s)}{\sqrt{r(s)}}$ is a continuous martingale with quadratic variation $\langle W(t), W(t)\rangle = \int_0^t ds = t$, hence, a one dimensional Brownian motion. Thus the process $r(t)$ satisfies the SDE

$$dr(t) \;=\; \left(\tfrac{n\sigma^2}{4} - \alpha r(t)\right)dt + \sigma\sqrt{r(t)}\,dW(t) \tag{21.8}$$

Definition 21.1: A Cox-Ingersoll-Ross process is a process r_t defined by the SDE

$$dr_t \;=\; (a - br_t)dt + \sigma\sqrt{r_t}\,dB_t \tag{21.9}$$

where $a, b, \sigma > 0$ are positive constants.

Theorem 21.2: Let r_t be a Cox-Ingersoll-Ross process starting at $r_0 = x$. Then

a) For $\lambda, \mu > 0$

$$\mathsf{E}\Big[e^{\lambda r_T - \mu\int_0^T r_s ds}\Big] \;=\; e^{-a\phi_{\lambda,\mu}(T) - x\psi_{\lambda,\mu}(T)} \tag{21.10}$$

where

$$\phi_{\lambda,\mu}(t) \;=\; -\frac{2}{\sigma^2}\log\left[\frac{2\gamma\, e^{\frac{t(b+\gamma)}{2}}}{\sigma^2\lambda(e^{\gamma t}-1)+\gamma-b+e^{\gamma t}(\gamma+b)}\right] \tag{21.11}$$

$$\psi_{\lambda,\mu}(t) \;=\; \frac{\lambda(\gamma+b)+e^{\gamma t}(\gamma-b)+2\mu(e^{\gamma t}-1)}{\sigma^2\lambda(e^{\gamma t}-1)+\gamma-b+e^{\gamma t}(\gamma+b)} \tag{21.12}$$

and $\gamma = \sqrt{b^2+2\sigma^2\mu}$.
b) For all $r_0 = x > 0$, the following holds: For $a < \sigma^2/2$,

$$\mathsf{P}\big(\text{there are infinitely many times for which } r_t = 0\,\big) \;=\; 1\,. \tag{21.13}$$

For $a \geq \sigma^2/2$,

$$\mathsf{P}\big(\text{there is at least one time for which } r_t = 0\,\big) \;=\; 0\,. \tag{21.14}$$

Proof: Part (a) can be obtained from (16.33), which reduced the left hand side of (21.10) to an expectation for a squared Bessel process, and the corresponding result (20.25). For a direct proof, using martingale methods as in the proof of (20.25), one may look in [4]. Part (b) is proven in exercise 6. ■

Finally we write down a complex version of (21.10) which will be used in the Heston model. It reads as follows.

Corollary 21.3: Let Y_t a Cox-Ingersoll-Ross process given by the SDE

$$dY_t \;=\; \kappa(\bar{y}-Y_t)dt + \sigma\sqrt{Y_t}\,dB_t \tag{21.15}$$

with initial condition $Y_0 = y$ and positive constants $\kappa, \bar{y}, \sigma > 0$. Let $\lambda = a+ib$, $\mu = c+id$. Then

$$\mathsf{E}\big[e^{\frac{\lambda}{2}Y_t - \frac{\mu}{2}\int_0^t Y_s ds}\big] \;=\; e^{\frac{\kappa+2f'/f(0)}{\sigma^2}y + \frac{\kappa^2}{\sigma^2}\bar{y}t}\; e^{\frac{2\kappa\bar{y}}{\sigma^2}\left\{\log[\frac{r_f(t)}{r_f(0)}]+i\left(\varphi_f(t)-\varphi_f(0)\right)\right\}} \tag{21.16}$$

where the function f is given by

$$\begin{aligned} f(s) \;&=\; \sqrt{\xi}\cosh[\sqrt{\xi}(t-s)] - \gamma\sinh[\sqrt{\xi}(t-s)] \\ &=:\; r_f(s)\, e^{i\varphi_f(s)} \end{aligned} \tag{21.17}$$

with a differentiable φ_f and

$$\gamma = \frac{\lambda\sigma^2}{4} - \frac{\kappa}{2} \tag{21.18}$$

$$\xi = \frac{\mu\sigma^2}{4} + \frac{\kappa^2}{4}\,. \tag{21.19}$$

Proof: The expectation can be reduced to an expectation involving a Bessel process. Recall formula (16.33),

$$\mathsf{E}\Big[e^{\frac{\lambda}{2}Y_t - \frac{\mu}{2}\int_0^t Y_s ds}\Big] = e^{\frac{\kappa}{\sigma^2}(y+\kappa\bar{y}t)}\ \mathsf{E}\Big[e^{(\frac{\lambda\sigma^2}{8}-\frac{\kappa}{4})X_t - (\frac{\mu\sigma^2}{8}+\frac{\kappa^2}{8})\int_0^t X_s ds}\Big] \tag{21.20}$$

where X_t is a squared Bessel process satisfying the SDE

$$dX_t = \frac{4\kappa\bar{y}}{\sigma^2}\,dt + 2\sqrt{X_t}\,dx_t \tag{21.21}$$

with initial condition $X_0 = \frac{4y}{\sigma^2} =: x$. Thus, by Corollary 16.3 we obtain

$$\begin{aligned}\mathsf{E}\Big[e^{\frac{\lambda}{2}Y_t - \frac{\mu}{2}\int_0^t Y_s ds}\Big] &= e^{\frac{\kappa}{\sigma^2}(y+\kappa\bar{y}t)}\ e^{\frac{f'(0)}{f(0)}\frac{x}{2} + \frac{2\kappa\bar{y}}{\sigma^2}\left\{\log[\frac{r_f(t)}{r_f(0)}] + i\left(\varphi_f(t)-\varphi_f(0)\right)\right\}} \\ &= e^{\frac{\kappa+2f'/f(0)}{\sigma^2}y + \frac{\kappa^2}{\sigma^2}\bar{y}t}\ e^{\frac{2\kappa\bar{y}}{\sigma^2}\left\{\log[\frac{r_f(t)}{r_f(0)}] + i\left(\varphi_f(t)-\varphi_f(0)\right)\right\}}\end{aligned} \tag{21.22}$$

where the function f is given by

$$f(s) = \sqrt{\xi}\cosh[\sqrt{\xi}(t-s)] - \gamma\sinh[\sqrt{\xi}(t-s)] \tag{21.23}$$

with $\gamma = \frac{\lambda\sigma^2}{4} - \frac{\kappa}{2}$ and $\xi = \frac{\mu\sigma^2}{4} + \frac{\kappa^2}{4}$. ■