Kapitel 8: Stochastic Calculus und Payoff Replication im Black-Scholes Modell

In the last chapter we derived the Black-Scholes equation by considering the recursion relations of the replicating portfolio in the approximating Binomial model and then we took the continuous time limit. In this chapter we ask the following question: Is it possible to derive the Black-Scholes equation directly from the Black-Scholes model

$$dS_t/S_t \;=\; \mu\, dt + \sigma\, dx_t \tag{1}$$

in continuous time, without using the approximating Binomial model? The answer is yes. In the following, we will use the Ito-formula to make the appropriate calculations. For simplicity, we start with zero rates, $r = 0$.

In the first chapter we saw that the portfolio value V_{t_k} of a selffinancing strategy, which holds $\delta_{t_{k-1}}$ stocks 'at the end of time t_{k-1}' or 'at the beginning of time t_k' and readjusts this to δ_{t_k} stocks 'at the end of time t_k after the asset price has switched from $S_{t_{k-1}}$ to S_{t_k}, is given by

$$V_{t_k} \;=\; V_0 + \sum_{j=1}^{k} \delta_{t_{j-1}} \cdot (S_{t_j} - S_{t_{j-1}}) \;=\; V_{t_{k-1}} + \delta_{t_{k-1}} \cdot (S_{t_k} - S_{t_{k-1}}) \tag{2}$$

In continuous time with 'continuous trading' this may be rewritten as a stochastic integral, as an Ito-integral

$$V_t \;=\; V_0 \;+\; \int_0^t \delta_\tau \, dS_\tau \tag{3}$$

or in differential form, if we subtract the $V_{t_{k-1}}$-term on the right hand side (2),

$$dV \;=\; \delta\, dS \tag{4}$$

where dV is the limit of

$$V_t(S_t) - V_{t-\Delta t}(S_{t-\Delta t}) \;=\; V(S_t, t) - V(S_{t-\Delta t}, t - \Delta t) \;\stackrel{\Delta t \to 0}{\rightarrow}\; dV \tag{5}$$

Since we have $V = V(S_t, t)$ and S_t is a stochastic quantity, we have to use the Ito-Formula, the differential version of the Ito-Formula, to calculate the dV. Let's start by recalling the calculation rules for the Brownian motion,

$$\begin{aligned} (dx_t)^2 &= dt \\ dx_t\, dt &= 0 \\ (dt)^2 &= 0 \end{aligned} \tag{6}$$

As a consequence of these rules, we obtained the Ito-Formula in Chapter 4. There was a differential version and an integral version. Let's summarize both versions in the following

Theorem 8.1 (Ito-Formula for Functions of a Brownian Motion): Let

$$F \;=\; F(x) \;\;:\;\; \mathbb{R} \,\to\, \mathbb{R}$$

be an arbitrary two-times differentiable function of one variable and let $\{x_t\}_{0\leq t\leq T}$ be a Brownian motion. Then we have the following identities:

a) Differential Version: Let $dF(x_t) := F(x_t) - F(x_{t-dt})$. Then

$$\begin{aligned} dF(x_t) \;&=\; F'(x_t)\,dx_t \;+\; \tfrac{1}{2}\,F''(x_t)\,(dx_t)^2 \\ &=\; F'(x_t)\,dx_t \;+\; \tfrac{1}{2}\,F''(x_t)\,dt \end{aligned}$$

b) Integral Version: We have

$$F(x_T) \;-\; F(x_0) \;=\; \int_0^T F'(x_t)\,dx_t \;+\; \tfrac{1}{2}\int_0^T F''(x_t)\,dt$$

where the stochastic dx_t-integral above is to be defined as an Ito-integral according to

$$\int_0^T f(x_t)\,dx_t \;=\; \lim_{\Delta t\to 0}\sum_{k=1}^{N} f(x_{t_{k-1}})\,\Delta x_{t_k} \;=\; \lim_{\Delta t\to 0}\sum_{k=1}^{N} f(x_{t_{k-1}})\,\sqrt{\Delta t}\,\phi_k$$

and the Brownain motion $x_{t_{k-1}}$ at time $t_{k-1} = (k-1)\Delta t$ given by

$$x_{t_{k-1}} \;=\; \sqrt{\Delta t}\,\sum_{j=1}^{k-1}\phi_j \;\;.$$

A slightly generalized version of this is the following

Theorem 8.2 (Ito-Formula for Functions of a Brownian Motion and Time): Let

$$F \;=\; F(x,t) \;\;:\;\; \mathbb{R}^2 \,\to\, \mathbb{R}$$

be an arbitrary two-times differentiable function of two variables and let $\{x_t\}_{0\leq t\leq T}$ be a Brownian motion. Then we have the following identities:

a) Differential Version: Let $dF(x_t,t) := F(x_t,t) - F(x_{t-dt}, t-dt)$. Then

$$\begin{aligned} dF \;&=\; \tfrac{\partial F}{\partial x}\,dx_t \;+\; \tfrac{1}{2}\,\tfrac{\partial^2 F}{\partial x^2}\,(dx_t)^2 \;+\; \tfrac{\partial F}{\partial t}\,dt \\ &=\; \tfrac{\partial F}{\partial x}\,dx_t \;+\; \Big\{\tfrac{1}{2}\,\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t}\Big\}\,dt \end{aligned}$$

b) Integral Version: We have

$$F(x_T,T) \;-\; F(x_0,0) \;=\; \int_0^T \tfrac{\partial F}{\partial x}\,dx_t \;+\; \int_0^T\Big\{\tfrac{1}{2}\,\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t}\Big\}\,dt$$

where the stochastic dx_t-integral above again is to be defined as an Ito-integral.

Since we want to calculate quantities like

$$V_t(S_t) - V_{t-\Delta t}(S_{t-\Delta t}) \;=\; V(S_t,t) - V(S_{t-\Delta t}, t-\Delta t) \;\stackrel{\Delta t\to 0}{\to}\; dV$$

where S is given by

$$S \;=\; S(x_t,t) \;=\; S_0\, e^{\sigma x_t + (\mu-\sigma^2/2)t}$$

we need actually a more general version than the two theorems above. Namely, the F above in the theorems is now the V, the portfolio value. However, we want to consider the V as a function of S_t, not of x_t. That is, we plug in stochastic objects, but not directly the Brownian motion, but functions of it. To specify the class of stochastic objects we can plug into the V or some $F = F(S_t,t)$, we need the following

Definition 8.3: An Ito diffusion is a stochastic process X_t given by the SDE

$$dX_t \;=\; a(X_t,t)\,dt \;+\; b(X_t,t)\,dx_t$$

with x_t being a Brownian motion.

Example: The Black-Scholes model given by the geometric Brownain motion

$$S_t \;=\; S(x_t,t) \;=\; S_0\, e^{\sigma x_t + (\mu-\sigma^2/2)t}$$

is an Ito-diffusion since with Theorem 8.2

$$\begin{aligned} dS_t \;&=\; \tfrac{\partial S}{\partial x}\,dx_t \;+\; \Big\{ \tfrac{1}{2}\tfrac{\partial^2 S}{\partial x^2} \;+\; \tfrac{\partial S}{\partial t} \Big\}\,dt \\ &=\; \sigma\, S_t\, dx_t \;+\; \Big\{ \tfrac{\sigma^2}{2}\, S_t \;+\; (\mu - \sigma^2/2)\, S_t \Big\}\,dt \\ &=\; \sigma\, S_t\, dx_t \;+\; \mu\, S_t\, dt \end{aligned}$$

which is of course equivalent to the SDE we derived already in Chapter 4,

$$dS_t/S_t \;=\; \mu\, dt \;+\; \sigma\, dx_t$$

Thus we have

$$\begin{aligned} a(S_t,t) \;&=\; \mu\, S_t \\ b(S_t,t) \;&=\; \sigma\, S_t \end{aligned}$$

in Definition 8.3 and S_t is an Ito-diffusion. Now we can state a third theorem which summarizes the formulae we will actually use:

Theorem 8.4 (Ito-Formula for Functions of an Ito-Diffusion and Time): Let

$$F \;=\; F(x,t) \;:\; \mathbb{R}^2 \;\to\; \mathbb{R}$$

be an arbitrary two-times differentiable function of two variables and let $\{x_t\}_{0\le t\le T}$ be a Brownian motion. Let X_t be an Ito-diffusion given by the SDE

$$dX_t \;=\; a(X_t,t)\,dt \;+\; b(X_t,t)\,dx_t$$

We plug X_t into the first argument of F and consider the function $F = F(X_t,t)$. Then we have the following identities:

a) Differential Version: Let $dF(X_t, t) := F(X_t, t) - F(X_{t-dt}, t - dt)$ with X_t being the Ito-diffusion from above. Then

$$\begin{aligned} dF &= \tfrac{\partial F}{\partial x}\, dX_t \;+\; \tfrac{1}{2}\,\tfrac{\partial^2 F}{\partial x^2}\,(dX_t)^2 \;+\; \tfrac{\partial F}{\partial t}\, dt \\ &= \tfrac{\partial F}{\partial x}\,(a\, dt \;+\; b\, dx_t) \;+\; \tfrac{1}{2}\,\tfrac{\partial^2 F}{\partial x^2}\,(a\, dt \;+\; b\, dx_t)^2 \;+\; \tfrac{\partial F}{\partial t}\, dt \\ &= \tfrac{\partial F}{\partial x}\,(a\, dt \;+\; b\, dx_t) \;+\; \tfrac{b^2}{2}\,\tfrac{\partial^2 F}{\partial x^2}\, dt \;+\; \tfrac{\partial F}{\partial t}\, dt \\ &= \Big\{ a\,\tfrac{\partial F}{\partial x} \;+\; \tfrac{b^2}{2}\,\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t} \Big\}\, dt \;+\; b\,\tfrac{\partial F}{\partial x}\, dx_t \end{aligned}$$

b) Integral Version: We have

$$F(X_T, T) \;-\; F(X_0, 0) \;=\; \int_0^T \Big\{ a\,\tfrac{\partial F}{\partial x} \;+\; \tfrac{b^2}{2}\,\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t} \Big\}\, dt \;+\; \int_0^T b\,\tfrac{\partial F}{\partial x}\, dx_t$$

where the stochastic dx_t-integral above again is to be defined as an Ito-integral.

Now we are in a position to calculate dV, the change of the value of the replicating portfolio in contiuous time. With the Ito-Formula, we get

$$\begin{aligned} dV &= \tfrac{\partial V}{\partial S}\, dS \;+\; \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\,(dS)^2 \;+\; \tfrac{\partial V}{\partial t}\, dt \\ &= \tfrac{\partial V}{\partial S}\, dS \;+\; \Big\{ \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \Big\}\, dt \end{aligned} \tag{7}$$

Thus, if this change should be given by trading δ stocks of the underlying, that is, if this should be equal to $\delta\, dS$,

$$dV \;=\; \tfrac{\partial V}{\partial S}\, dS \;+\; \Big\{ \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \Big\}\, dt \;\stackrel{!}{=}\; \delta\, dS \tag{8}$$

we have to have the equations

$$\delta \;=\; \tfrac{\partial V}{\partial S} \tag{9}$$

which coincides with the definition of the previous chapter and

$$\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \;=\; 0 \tag{10}$$

which is the Black-Scholes equation for zero interest rates. Thus, if (9) and (10) are fulfilled, we can use the integral version of Theorem 8.4 with $X_t = S_t$ and $F(X_t, t) = V(S_t, t)$ and

$$(dS_t)^2 \;=\; S_t^2\,(\mu\, dt \;+\; \sigma\, dx_t)^2 \;\stackrel{\text{Rechenregeln BB}}{=}\; S_t^2\,\sigma^2\, dt \tag{11}$$

to obtain

$$\begin{aligned} V(S_T, T) \;-\; V(S_0, 0) &= \int_0^T \tfrac{\partial V}{\partial S}\, dS_t \;+\; \int_0^T \underbrace{\Big\{ \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \Big\}}_{=\,0}\, dt \\ &= \int_0^T \delta(S_t, t)\, dS_t \end{aligned} \tag{12}$$

Thus, some payoff $H = H(S_T)$ can be exactly replicated in continuous time if we impose the final condition

$$V(S_T, T) \;=\; H(S_T) \tag{13}$$

in addition to (10).

When interest rates are present, a similar derivation can be done. Since this is an important calculation and an important result, in the continuous time Black-Scholes model exact payoff replication is still possible, we state this in a separate theorem.

Theorem 8.5: Let $H = H(S)$ be some option payoff and let $V = V(S,t)$ be a solution of the Black-Scholes PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \\ V(S,T) &= H(S) \end{aligned}$$

and let $\{S_t\}_{t\geq 0}$ be a stochastic price process given by the Black-Scholes Modell with real world drift μ,

$$\frac{dS_t}{S_t} \;=\; \mu\, dt \,+\, \sigma\, dx_t$$

Furthermore let $V_0 = V(S_0, 0)$ be the option price of H and define a trading strategy by the following delta's,

$$\delta_t \;=\; \delta(S_t, t) \;:=\; \frac{\partial V(S_t, t)}{\partial S_t}$$

As usual, we use the notation with small letters

$$s_t \;:=\; e^{-rt} S_t$$

for the discounted price process. Then the following equation holds:

$$e^{-rT} H(S_T) \;=\; V_0 \,+\, \int_0^T \delta_t\, ds_t$$

This equation means (recall part (b) of Theorem 1.1 in the first chapter) that in the Black-Scholes model any option payoff $H = H(S_T)$ can be replicated with a suitable trading strategy.

Proof: Let

$$v(S,t) \;:=\; e^{-rt}\, V(S,t)$$

Then, since $v_0 = V_0$,

$$\begin{aligned} e^{-rT} H(S_T) - V_0 &= v(S_T, T) - v(S_0, 0) \\ &= \int_0^T dv \end{aligned}$$

where

$$\begin{aligned} dv &= d\big(e^{-rt}V\big) \\ &= d(e^{-rt})\,V \;+\; e^{-rt}\,dV \;+\; d(e^{-rt})\,dV \\ &= -r\,e^{-rt}\,dt\,V \;+\; e^{-rt}\,dV \;+\; 0 \end{aligned}$$

Now,

$$dV \;=\; \frac{\partial V}{\partial S}\,dS_t \;+\; \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2 \;+\; \frac{\partial V}{\partial t}\,dt$$

Using

$$\delta_t \;=\; \frac{\partial V}{\partial S}(S_t, t)$$

and, recalling the calculation rules for Brownian motion $(dx_t)^2 = dt$ and $dx_t\,dt = (dt)^2 = 0$,

$$\begin{aligned} (dS_t)^2 &= (\mu\,S_t\,dt + \sigma\,S_t\,dx_t)^2 \\ &= 0 + 0 + \sigma^2\,S_t^2\,(dx_t)^2 \\ &= \sigma^2\,S_t^2\,dt \end{aligned}$$

we obtain

$$dV \;=\; \delta_t\,dS_t \;+\; \left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt$$

We have to express dS_t through ds_t where $s_t = e^{-rt}\,S_t$. We have

$$\begin{aligned} ds_t &= d(e^{-rt}S_t) \\ &= -r\,e^{-rt}\,dt\,S_t \;+\; e^{-rt}\,dS_t \end{aligned}$$

or

$$e^{-rt}\,dS_t \;=\; ds_t \;+\; r\,e^{-rt}\,dt\,S_t$$

which gives

$$\begin{aligned} e^{-rt}\,dV &= \delta_t\,e^{-rt}\,dS_t \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt \\ &= \delta_t\left[ds_t \;+\; r\,e^{-rt}\,dt\,S_t\right] \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt \\ &= \delta_t\,ds_t \;+\; rS_t\frac{\partial V}{\partial S}\,dt \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt \end{aligned}$$

Thus,

$$\begin{aligned} dv &= -r\,e^{-rt}\,dt\,V \;+\; e^{-rt}\,dV \\ &= -r\,e^{-rt}\,dt\,V \;+\; \delta_t\,ds_t \;+\; rS_t\frac{\partial V}{\partial S}\,dt \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt \\ &= \delta_t\,ds_t \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; rS_t\frac{\partial V}{\partial S} \;+\; \frac{\partial V}{\partial t} \;-\; rV\right\}dt \\ &= \delta_t\,ds_t \end{aligned}$$

where we used the Black-Scholes PDE in the last line. Thus,

$$\begin{aligned} e^{-rT} H(S_T) \; - \; V_0 &= v(S_T, T) \; - \; v(S_0, 0) \\ &= \int_0^T dv \\ &= \int_0^T \delta_t \, ds_t \end{aligned}$$

and the theorem is proven. ∎

Excel/VBA-Simulation: Wir wollen jetzt die Aussage von Theorem 8.5 durch eine geeignete Excel/VBA-Simulation verifizieren. Der Einfachheit halber wollen wir annehmen, dass die Zinsen 0 sind, dann müssen wir also nicht zwischen diskontierten und undiskontierten Grössen unterscheiden, wir haben $s_t = S_t$. Weiterhin wollen wir eine standard Call-Option mit payoff

$$H_{\text{call}} \;=\; \max\{\, S_T - K \,,\, 0 \,\}$$

betrachten, für die wir sowohl für den Preis als auch für das Delta bereits eine analytische Formel hergeleitet haben, im Kapitel 6. Und zwar hatten wir

$$V_{\text{call},t} \;=\; S_t \, N(d_+) \;-\; K \, e^{-r(T-t)} \, N(d_-)$$

mit

$$d_\pm \;:=\; \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

und die Formel für das Delta δ_t zur Zeit t war einfach

$$\delta_{\text{call}} \;=\; N(d_+)$$

Ok, machen wir jetzt die Implementation: Wir müssen die Grösse

$$V(t_N) \;=\; V_0 \;+\; \sum_{k=1}^{N} \delta(t_{k-1}) \times \big(S_{t_k} \;-\; S_{t_{k-1}} \big)$$

berechnen mit

$$\delta(t_j) \;=\; N\big[\, d_+(t_j) \,\big]$$

und

$$d_+(t_j) \;:=\; \frac{\log \frac{S_{t_j}}{K} + (r + \frac{\sigma^2}{2})(t_N - t_j)}{\sigma\sqrt{t_N - t_j}}$$

und dann sollte in etwa gelten (exakt im Limes $\Delta t \to 0$)

$$V(t_N) \;\approx\; H(S_{t_N}) \;=\; \max\{\, S_{t_N} - K \,,\, 0 \,\}$$

für beliebige Black-Scholes Pfade gegeben durch

$$S_{t_k} \;=\; S_{t_{k-1}} \left(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\, \phi_k \right)$$

Man beachte, dass diese Pfade einen beliebigen real world Drift μ haben können, da muss nicht der Zinssatz r stehen, den wir hier ja auf 0 setzen wollen. Das μ ist also ein weiterer Input-Parameter für die Excel-Simulation.