

Chapter 11

Barrier and Lookback Options in the Black-Scholes Model

We now have all ingredients to derive an analytical formula for the price of a barrier option in the Black-Scholes model. There is the following

Theorem 11.1: Consider the barrier option with payoff

$$H(S_T) = \max\{S_T - K, 0\} \chi\left(\min_{0 \leq t \leq T} S_t > B\right), \quad K \geq B > 0 \quad (11.1)$$

a ‘down-and-out’ Barrier-Call with strike K , barrier B and maturity T . Observe that this is only a reasonable payoff if $S_0 > B$. Let the asset price dynamics be given by a Black-Scholes model with drift μ and volatility σ ,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dx_t$$

or equivalently

$$S_t = S_t^{(\mu)} = S_0 e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t}$$

Let further be $V_{\text{Call}}(S, t)$ be the time- t Black-Scholes price of the call $\max\{S_T - K, 0\}$ given by (6.12) of Theorem 6.1, and let $\kappa := \frac{2r}{\sigma^2}$. Then the time- t price of the barrier option (11.1) is given by

$$V_{\text{Barrier}}(S, t) = V_{\text{Call}}(S, t) - \left(\frac{S}{B}\right)^{1-\kappa} V_{\text{Call}}\left(\frac{B^2}{S}, t\right) \quad (11.2)$$

where $S = S_t$ is the underlying asset price at time t (which is known at time t).

Proof: Recall that $c = \frac{r}{\sigma} - \frac{\sigma}{2}$. From (10.7), (10.10) and (10.23) we get

$$V_{\text{Barrier}}(S_0, t=0) = e^{-rT} \int H(S_0 e^{\sigma y_T}, S_0 e^{\sigma \min_{0 \leq t \leq T} \{y_t\}}) e^{cy_T - \frac{c^2}{2}T} dW(\{y_t\}_{0 < t \leq T})$$

$$\begin{aligned}
&= e^{-rT} \int_{\mathbb{R}^2} (S_0 e^{\sigma a} - K)_+ \chi(S_0 e^{\sigma b} > B) e^{ca - \frac{c^2}{2}T} \rho_{y_T, \min_{0 \leq t \leq T}\{y_t\}}(a, b) da db \\
&= e^{-rT} \int_{\mathbb{R}^2} (S_0 e^{\sigma a} - K)_+ \chi(S_0 e^{\sigma b} > B) e^{ca - \frac{c^2}{2}T} \left(-\frac{2}{T}\right) \varphi'\left(\frac{a-2b}{\sqrt{T}}\right) \chi(a > b) da db \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a} - K)_+ e^{ca - \frac{c^2}{2}T} \int_{\mathbb{R}} db \chi\left(a > b > -\frac{1}{\sigma} \log\left(\frac{S_0}{B}\right)\right) \left(-\frac{2}{T}\right) \varphi'\left(\frac{a-2b}{\sqrt{T}}\right) \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a} - K)_+ e^{ca - \frac{c^2}{2}T} \chi\left(a > -\frac{1}{\sigma} \log\left(\frac{S_0}{B}\right)\right) \frac{1}{\sqrt{T}} \varphi\left(\frac{a-2b}{\sqrt{T}}\right) \Big|_{-\frac{1}{\sigma} \log\left(\frac{S_0}{B}\right)}^a \quad (11.3) \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a} - K)_+ e^{ca - \frac{c^2}{2}T} \frac{1}{\sqrt{T}} \left[\varphi\left(-\frac{a}{\sqrt{T}}\right) - \varphi\left(\frac{a + \frac{2}{\sigma} \log\left(\frac{S_0}{B}\right)}{\sqrt{T}}\right) \right] \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a} - K)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{(a-cT)^2}{2T}} \\
&\quad - e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a} - K)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{\left(a + \frac{2}{\sigma} \log\left(\frac{S_0}{B}\right)\right)^2}{2T}} e^{ca - \frac{c^2}{2}T} \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a + \sigma cT} - K)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{a^2}{2T}} \\
&\quad - e^{-rT} \left(\frac{S_0}{B}\right)^{-\frac{2c}{\sigma}} \int_{\mathbb{R}} da (S_0 e^{\sigma a - 2 \log\left(\frac{S_0}{B}\right)} - K)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{a^2}{2T}} e^{ca - \frac{c^2}{2}T} \\
&= e^{-rT} \int_{\mathbb{R}} da (S_0 e^{\sigma a + (r - \frac{\sigma^2}{2})T} - K)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{a^2}{2T}} \\
&\quad - e^{-rT} \left(\frac{S_0}{B}\right)^{-(\kappa-1)} \int_{\mathbb{R}} da \left(\frac{B^2}{S_0} e^{\sigma a} - K\right)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{(a-cT)^2}{2T}} \\
&= V_{\text{Call}}(S_0, 0) - \left(\frac{S}{B}\right)^{-(\kappa-1)} V_{\text{Call}}\left(\frac{B^2}{S_0}, 0\right) \quad (11.4)
\end{aligned}$$

Observe that in (11.3) we used that $K \geq B$ since otherwise a may be less than $\frac{1}{\sigma} \log \frac{B}{S_0}$. This proves the theorem for $t = 0$. For general $t \in [0, T]$, the result follows from the following lemma. ■

Lemma 11.2: Let $dW_{(t, x_t)}^{(t, T]}$ be the Wiener measure for the Brownian motion on the interval $[t, T]$ starting at x_t at time t . Let

$$S_t^{(\mu)} = S_0 e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \quad (11.5)$$

be a geometric Brownian motion and let $d\tilde{W}_{(t, x_t)}^{(t, T]}$ be the equivalent martingale measure defined by the kernels \tilde{p} of (9.14). Let $H(\{S_s^{(\mu)}\}_{0 \leq s \leq T}) = H(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_s^{(\mu)}\}_{t < s \leq T})$ be some payoff. Then

$$\begin{aligned}
&\int H(\{S_s^{(\mu)}\}_{0 \leq s \leq T}) d\tilde{W}_{(t, x_t)}^{(t, T]}(\{x_s\}_{t < s \leq T}) \\
&= \int H(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_t^{(\mu)} e^{\sigma y_s + (r - \frac{\sigma^2}{2})s}\}_{0 < s \leq T-t}) dW_{(0, 0)}^{(0, T-t]}(\{y_s\}_{0 < s \leq T-t}) \quad (11.6)
\end{aligned}$$

Observe that in this formula the asset prices $\{S_s^{(\mu)}\}_{0 \leq s \leq t}$ are all known, they have all realized already, since current time is t and not 0.

Proof: As in Theorem 9.2 we have

$$\begin{aligned}
& \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq T}\right) d\tilde{W}_{(t,x_t)}^{(t,T]}(\{x_s\}_{t < s \leq T}) \\
&= \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_s^{(\mu)}\}_{t < s \leq T}\right) d\tilde{W}_{(t,x_t)}^{(t,T]}(\{x_s\}_{t < s \leq T}) \\
&= \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_s^{(r)}\}_{t < s \leq T}\right) dW_{(t,x_t + \frac{\mu-r}{\sigma}t)}^{(t,T]}(\{\tilde{x}_s\}_{t < s \leq T}) \tag{11.7}
\end{aligned}$$

For $s \in (t, T]$ we have with the definition $y_u := \tilde{x}_{u+t} - x_t - \frac{\mu-r}{\sigma}t$, $u > 0$,

$$\begin{aligned}
S_s^{(r)} &= S_0 e^{\sigma \tilde{x}_s + (r - \frac{\sigma^2}{2})s} \\
&= S_0 e^{\sigma(\tilde{x}_{s-t+t} - x_t - \frac{\mu-r}{\sigma}t) + \sigma x_t + (\mu-r)t + (r - \frac{\sigma^2}{2})s} \\
&= S_0 e^{\sigma y_{s-t} + \sigma x_t + (\mu - \frac{\sigma^2}{2})t + (r - \frac{\sigma^2}{2})(s-t)} \\
&= S_t^{(\mu)} e^{\sigma y_{s-t} + (r - \frac{\sigma^2}{2})(s-t)} \tag{11.8}
\end{aligned}$$

The Wiener measure in the last line of (11.7) is given by ($s_j = j\Delta t$, $N_t = t/\Delta t$)

$$dW_{(t,x_t + \frac{\mu-r}{\sigma}t)}^{(t,T]}(\{\tilde{x}_s\}_{t < s \leq T}) = \lim_{\Delta t \rightarrow 0} \prod_{j=N_t+1}^{N_T} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(\tilde{x}_{s_j} - \tilde{x}_{s_{j-1}})^2}{2\Delta t}} d\tilde{x}_{s_j} \tag{11.9}$$

Observe that on the right hand side of (11.9) the variables $\tilde{x}_{s_{N_t+1}}, \dots, \tilde{x}_{s_{N_T}}$ are simply integration variables whereas $\tilde{x}_{s_{N_t}} = \tilde{x}_t := x_t + \frac{\mu-r}{\sigma}t$ is the starting point of the Brownian motion. Because this starting point is not x_t but $x_t + \frac{\mu-r}{\sigma}t$, we used the tilde for the integration variables. x_t is related to the price process according to (11.5). Making the substitution of variables, for $t < s \leq T$,

$$\begin{aligned}
y_{s-t} &:= \tilde{x}_s - \tilde{x}_t \\
&= \tilde{x}_s - x_t - \frac{\mu-r}{\sigma}t \\
dy_{s-t} &= d\tilde{x}_s \tag{11.10}
\end{aligned}$$

we obtain ($y_0 := 0$)

$$\begin{aligned}
dW_{(t,x_t + \frac{\mu-r}{\sigma}t)}^{(t,T]}(\{\tilde{x}_s\}_{t < s \leq T}) &= \lim_{\Delta t \rightarrow 0} \prod_{j=N_t+1}^{N_T} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(y_{s_j-t} - y_{s_{j-1}-t})^2}{2\Delta t}} dy_{s_j-t} \\
&= \lim_{\Delta t \rightarrow 0} \prod_{j=1}^{N_T-N_t} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(y_{s_j} - y_{s_{j-1}})^2}{2\Delta t}} dy_{s_j} \\
&= dW_{(0,0)}^{(0,T-t]}(\{y_s\}_{0 < s \leq T-t}) \tag{11.11}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq T}\right) d\tilde{W}_{(t,x_t)}^{(t,T]}(\{x_s\}_{t < s \leq T}) \\
&= \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_s^{(r)}\}_{t < s \leq T}\right) dW_{(t,x_t + \frac{\mu-r}{\sigma}t)}^{(t,T]}(\{\tilde{x}_s\}_{t < s \leq T}) \\
&= \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_t^{(\mu)} e^{\sigma y_{s-t} + (r - \frac{\sigma^2}{2})(s-t)}\}_{t < s \leq T}\right) dW_{(t,x_t + \frac{\mu-r}{\sigma}t)}^{(t,T]}(\{\tilde{x}_s\}_{t < s \leq T}) \\
&= \int H\left(\{S_s^{(\mu)}\}_{0 \leq s \leq t}, \{S_t^{(\mu)} e^{\sigma y_s + (r - \frac{\sigma^2}{2})s}\}_{0 < s \leq T-t}\right) dW_{(0,0)}^{(0,T-t]}(\{y_s\}_{0 < s \leq T-t}) \quad (11.12)
\end{aligned}$$

which proves the lemma. ■

Lookback Options

We compute the price of a lookback put with payoff

$$H(\{S_s\}_{0 \leq s \leq T}) = \max_{0 \leq s \leq T} \{S_s\} - S_T \quad (11.13)$$

where $S_s = S_s^{(\mu)} = S_0 e^{\sigma x_s + (\mu - \frac{\sigma^2}{2})s}$ is a geometric Brownian motion. The price at time $t \in (0, T)$ is given by

$$V_t = e^{-r(T-t)} \mathbf{E}_{\tilde{W}_{(t,x_t)}} \left[\max_{0 \leq s \leq T} S_s - S_T \right] \quad (11.14)$$

Let

$$M_t := \max_{0 \leq s \leq t} S_s \quad (11.15)$$

There is the following

Theorem 11.3: Let $\kappa = \frac{2r}{\sigma^2}$. Then the time- t price V_t of the lookback put (11.13) is given by

$$\begin{aligned}
V_t = e^{-r\tau} & \left\{ M_t N(d_1) + e^{r\tau} S_t N(d_{0,+}) \right. \\
& \left. + \frac{1}{\kappa} \left[e^{r\tau} S_t N(d_{0,+}) - S_t^{1-\kappa} M_t^\kappa N(d_{0,-}) \right] \right\} - S_t \quad (11.16)
\end{aligned}$$

Here $M_t := \max_{0 \leq s \leq t} S_s$ is the current realized maximum up to time t , $\tau = T - t$ is the remaining time to maturity and the d 's are given by

$$\begin{aligned}
d_{0,\pm} &:= \frac{\log\left[\frac{S_t}{M_t}\right] + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}} \pm \frac{r}{\sigma}\sqrt{\tau} \\
d_1 &:= \frac{\log\left[\frac{M_t}{S_t}\right] + (\frac{\sigma^2}{2} - r)\tau}{\sigma\sqrt{\tau}}
\end{aligned}$$

Proof: Since $\max_{0 \leq s \leq T} S_s = \max\{M_t, \max_{t \leq s \leq T} S_s\}$ and because of the above lemma we have

$$V_t = e^{-r(T-t)} \mathbb{E}_{\tilde{W}_{(t,x_t)}^{(t,T)}} \left[\max\{M_t, \max_{t \leq s \leq T} S_s^{(\mu)}\} - S_t^{(\mu)} \right] \quad (11.17)$$

$$\begin{aligned} &= e^{-r(T-t)} \mathbb{E}_{W_{(0,0)}^{(0,T-t)}} \left[\max\{M_t, \max_{0 \leq s \leq T-t} S_t^{(\mu)} e^{\sigma y_s + (r - \frac{\sigma^2}{2})s}\} - S_t^{(\mu)} e^{\sigma y_{T-t} + (r - \frac{\sigma^2}{2})(T-t)} \right] \\ &= e^{-r(T-t)} \mathbb{E}_{W_{(0,0)}^{(0,T-t)}} \left[\max\{M_t, S_t^{(\mu)} e^{\sigma \max_{0 \leq s \leq T-t} \{y_s + (\frac{r}{\sigma} - \frac{\sigma}{2})s\}}\} - S_t^{(\mu)} e^{\sigma(y_{T-t} + (\frac{r}{\sigma} - \frac{\sigma}{2})(T-t))} \right] \end{aligned}$$

We abbreviate $\tau = T - t$, $S_t^{(\mu)} = S_t$ and make a Girsanov transformation $y_s + cs = v_s$, $c = \frac{r}{\sigma} - \frac{\sigma}{2}$. Then

$$\begin{aligned} V_t &= e^{-r\tau} \int \left[\max\{M_t, S_t e^{\sigma \max_{0 \leq s \leq \tau} \{v_s\}}\} - S_t e^{\sigma v_\tau} \right] e^{cy_\tau + \frac{c^2}{2}\tau} dW_{(0,0)}(\{v_s\}_{0 < s \leq \tau}) \\ &= e^{-r\tau} \int \left[\max\{M_t, S_t e^{\sigma \max_{0 \leq s \leq \tau} \{v_s\}}\} - S_t e^{\sigma v_\tau} \right] e^{cv_\tau - \frac{c^2}{2}\tau} dW_{(0,0)}(\{v_s\}_{0 < s \leq \tau}) \\ &= e^{-r\tau} \int_{\mathbb{R}^2} \left[\max\{M_t, S_t e^{\sigma b}\} - S_t e^{\sigma a} \right] e^{ca - \frac{c^2}{2}\tau} \times \\ &\quad \mathbb{P}\left(v_\tau \in [a, a + da), \max_{0 \leq s \leq \tau} \{v_s\} \in [b, b + db)\right) \\ &\stackrel{(10.23)}{=} e^{-r\tau} \int_{\mathbb{R}^2} \left[\max\{M_t, S_t e^{\sigma b}\} - S_t e^{\sigma a} \right] e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) \chi(a < b) da db \\ &= e^{-r\tau} S_t \int_{\mathbb{R}^2} \left[\max\{J_t/S_t, e^{\sigma b}\} - e^{\sigma a} \right] e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) \chi(a < b) da db \end{aligned} \quad (11.18)$$

We introduce $b_0 > 0$ according to

$$M_t/S_t = e^{\sigma b_0} \quad (11.19)$$

and write

$$\max\{M_t/S_t, e^{\sigma b}\} = M_t/S_t \chi(b < b_0) + e^{\sigma b} \chi(b \geq b_0) \quad (11.20)$$

This gives

$$\begin{aligned}
V_t &= e^{-r\tau} S_t \int_{\mathbb{R}^2} \left[M_t/S_t \chi(b < b_0) + e^{\sigma b} \chi(b \geq b_0) - e^{\sigma a} \right] \times \\
&\quad e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) \chi(a < b) da db \\
&= e^{-r\tau} S_t \left\{ \int_{\mathbb{R}^2} M_t/S_t \chi(a < b < b_0) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) da db \right. \\
&\quad + \int_{\mathbb{R}^2} e^{\sigma b} \chi(b \geq b_0) \chi(b > a) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) da db \\
&\quad \left. - \int_{\mathbb{R}^2} e^{\sigma a} \chi(b > a) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) da db \right\} \quad (11.21) \\
&=: e^{-r\tau} S_t \{I_1 + I_2 + I_3\}
\end{aligned}$$

We first perform the b -integrals. For I_1 , we obtain

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} M_t/S_t \chi(a < b_0) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{1}{\sqrt{\tau}}\right) \varphi \left(\frac{2b-a}{\sqrt{\tau}}\right) \Big|_a^{b_0} da db \\
&= \int_{\mathbb{R}} M_t/S_t \chi(a < b_0) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{1}{\sqrt{\tau}}\right) \left(\varphi \left(\frac{2b_0-a}{\sqrt{\tau}}\right) - \varphi \left(\frac{a}{\sqrt{\tau}}\right) \right) da db \\
&= M_t/S_t \int_{b_0}^{\infty} e^{c(2b_0-a') - \frac{c^2}{2}\tau} \left(-\frac{1}{\sqrt{\tau}}\right) \varphi \left(\frac{a'}{\sqrt{\tau}}\right) da' \\
&\quad - M_t/S_t \int_{-\infty}^{b_0} e^{ca - \frac{c^2}{2}\tau} \left(-\frac{1}{\sqrt{\tau}}\right) \varphi \left(\frac{a}{\sqrt{\tau}}\right) da \\
&= -M_t/S_t \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} e^{c(2b_0 - \sqrt{\tau}a) - \frac{c^2}{2}\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da \\
&\quad + M_t/S_t \int_{-\infty}^{\frac{b_0}{\sqrt{\tau}}} e^{c\sqrt{\tau}a - \frac{c^2}{2}\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da \\
&= -M_t/S_t e^{2cb_0} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a+\sqrt{\tau}c)^2}{2}} da + M_t/S_t \int_{-\infty}^{\frac{b_0}{\sqrt{\tau}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\sqrt{\tau}c)^2}{2}} da \\
&= -M_t/S_t (M_t/S_t)^{\frac{2c}{\sigma}} \left(1 - N\left(\frac{b_0+\tau c}{\sqrt{\tau}}\right)\right) + M_t/S_t N\left(\frac{b_0-\tau c}{\sqrt{\tau}}\right) \\
&= -(M_t/S_t)^{\kappa} \left(1 - N\left(\frac{b_0+\tau c}{\sqrt{\tau}}\right)\right) + M_t/S_t N\left(\frac{b_0-\tau c}{\sqrt{\tau}}\right) \quad (11.22)
\end{aligned}$$

where we introduced $\kappa = \frac{2c}{\sigma} + 1 = \frac{2r}{\sigma^2}$. The third term in (11.21) becomes

$$\begin{aligned}
I_3 &= - \int_{\mathbb{R}^2} e^{\sigma a} \chi(b > a) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) da db \\
&= - \int_{\mathbb{R}} e^{\sigma a} e^{ca - \frac{c^2}{2}\tau} \left(-\frac{1}{\sqrt{\tau}}\right) \varphi \left(\frac{2b-a}{\sqrt{\tau}}\right) \Big|_a^\infty da \\
&= - \int_{\mathbb{R}} e^{\sigma a} e^{ca - \frac{c^2}{2}\tau} \frac{1}{\sqrt{\tau}} \varphi \left(\frac{a}{\sqrt{\tau}}\right) da \\
&= - \int_{\mathbb{R}} e^{\sigma\sqrt{\tau}a} e^{c\sqrt{\tau}a - \frac{c^2}{2}\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da \\
&= -e^{\frac{(\sigma+c)^2}{2}\tau} e^{-\frac{c^2}{2}\tau} = -e^{r\tau}
\end{aligned} \tag{11.23}$$

since $(\sigma + c)^2 - c^2 = \sigma(\sigma + 2c) = \sigma(\sigma + \frac{2r}{\sigma} - \sigma) = 2r$. Finally,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} e^{\sigma b} \chi(b \geq b_0) \chi(b > a) e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) da db \\
&= \int_{b_0}^{\infty} db e^{\sigma b} \int_{-\infty}^b da e^{ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{2b-a}{\sqrt{\tau}}\right) \\
&= \int_{b_0}^{\infty} db e^{\sigma b} \int_b^{\infty} da e^{c(2b-a) - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{a}{\sqrt{\tau}}\right) \\
&= \int_{b_0}^{\infty} db \underbrace{e^{(\sigma+2c)b}}_{=u'} \underbrace{\int_b^{\infty} da e^{-ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{a}{\sqrt{\tau}}\right)}_{=v} \\
&= \frac{1}{\sigma+2c} e^{(\sigma+2c)b} \int_b^{\infty} da e^{-ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{a}{\sqrt{\tau}}\right) \Big|_{b_0}^{\infty} \\
&\quad + \int_{b_0}^{\infty} db \frac{1}{\sigma+2c} e^{(\sigma+2c)b} e^{-cb - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{b}{\sqrt{\tau}}\right) \\
&= -\frac{1}{\sigma+2c} e^{(\sigma+2c)b_0} \int_{b_0}^{\infty} da e^{-ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{a}{\sqrt{\tau}}\right) \\
&\quad + \int_{b_0}^{\infty} db \frac{1}{\sigma+2c} e^{(\sigma+c)b} e^{-\frac{c^2}{2}\tau} \left(-\frac{2}{\tau}\right) \varphi' \left(\frac{b}{\sqrt{\tau}}\right) \\
&= -\frac{1}{\sigma+2c} e^{(\sigma+2c)b_0} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da e^{-\sqrt{\tau}ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\sqrt{\tau}}\right) \varphi' (a) \\
&\quad + \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} db \frac{1}{\sigma+2c} e^{(\sigma+c)\sqrt{\tau}b} e^{-\frac{c^2}{2}\tau} \left(-\frac{2}{\sqrt{\tau}}\right) \varphi' (b)
\end{aligned} \tag{11.24}$$

Because of

$$\begin{aligned}
& \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da e^{-\sqrt{\tau}ca - \frac{c^2}{2}\tau} \left(-\frac{2}{\sqrt{\tau}}\right) \varphi'(a) \\
&= \frac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da a e^{-\sqrt{\tau}ca - \frac{c^2}{2}\tau} e^{-\frac{a^2}{2}} \\
&= \frac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da a e^{-\frac{(a+\sqrt{\tau}c)^2}{2}} \\
&= \frac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da (a + \sqrt{\tau}c) e^{-\frac{(a+\sqrt{\tau}c)^2}{2}} - \frac{2\sqrt{\tau}c}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da e^{-\frac{(a+\sqrt{\tau}c)^2}{2}} \\
&= \frac{2}{\sqrt{2\pi\tau}} e^{-\frac{(b_0+\sqrt{\tau}c)^2}{2\tau}} - 2c \left(1 - N\left(\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right)\right)
\end{aligned} \tag{11.25}$$

(11.24) becomes

$$\begin{aligned}
I_2 &= -\frac{1}{\sigma+2c} e^{(\sigma+2c)b_0} \left(\frac{2}{\sqrt{2\pi\tau}} e^{-\frac{(b_0+\sqrt{\tau}c)^2}{2\tau}} - 2c + 2cN\left(\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right) \right) \\
&\quad + \frac{1}{\sigma+2c} e^{\frac{(\sigma+c)^2-c^2}{2}\tau} \left(\frac{2}{\sqrt{2\pi\tau}} e^{-\frac{(b_0-\sqrt{\tau}(c+\sigma))^2}{2\tau}} + 2(c+\sigma) - 2(c+\sigma)N\left(\frac{b_0-\sqrt{\tau}(c+\sigma)}{\sqrt{\tau}}\right) \right) \\
&= \frac{2c}{\sigma+2c} e^{(\sigma+2c)b_0} \left(1 - N\left(\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right)\right) + \frac{2c+2\sigma}{\sigma+2c} e^{r\tau} \left(1 - N\left(\frac{b_0-\sqrt{\tau}(c+\sigma)}{\sqrt{\tau}}\right)\right)
\end{aligned} \tag{11.26}$$

Substituting this in (11.21), we arrive at

$$\begin{aligned}
V_t &= e^{-r\tau} S_t \left\{ -(M_t/S_t)^\kappa N\left(-\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right) + M_t/S_t N\left(\frac{b_0-\sqrt{\tau}c}{\sqrt{\tau}}\right) + \frac{2c}{\sigma+2c} e^{(\sigma+2c)b_0} N\left(-\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right) \right. \\
&\quad \left. + \frac{2c+2\sigma}{\sigma+2c} e^{r\tau} N\left(-\frac{b_0-\sqrt{\tau}(c+\sigma)}{\sqrt{\tau}}\right) - e^{r\tau} \right\} \\
&= e^{-r\tau} \left\{ -M_t^\kappa S_t^{1-\kappa} N\left(-\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right) + M_t N\left(\frac{b_0-\sqrt{\tau}c}{\sqrt{\tau}}\right) + \frac{2c+\sigma-\sigma}{\sigma+2c} S_t e^{(\sigma+2c)b_0} N\left(-\frac{b_0+\sqrt{\tau}c}{\sqrt{\tau}}\right) \right. \\
&\quad \left. + \frac{2c+\sigma+\sigma}{\sigma+2c} e^{r\tau} S_t N\left(-\frac{b_0-\sqrt{\tau}(c+\sigma)}{\sqrt{\tau}}\right) \right\} - S_t
\end{aligned} \tag{11.27}$$

Recall the definition of b_0 and c and κ ,

$$\begin{aligned}
M_t &= S_t e^{\sigma b_0} \\
c &= \frac{r}{\sigma} - \frac{\sigma}{2} \\
\kappa &= \frac{2r}{\sigma^2}
\end{aligned}$$

which gives $\sigma + 2c = \sigma\kappa$ and

$$e^{(\sigma+2c)b_0} = (e^{\sigma b_0})^\kappa = \left(\frac{M_t}{S_t}\right)^\kappa$$

and

$$S_t e^{(\sigma+2c)b_0} = S_t^{1-\kappa} M_t^\kappa$$

Thus we obtain from (18.23),

$$\begin{aligned} V_t &= e^{-r\tau} \left\{ -M_t^\kappa S_t^{1-\kappa} N\left(-\frac{b_0+\tau c}{\sqrt{\tau}}\right) + M_t N\left(\frac{b_0-\tau c}{\sqrt{\tau}}\right) + S_t^{1-\kappa} M_t^\kappa N\left(-\frac{b_0+\tau c}{\sqrt{\tau}}\right) \right. \\ &\quad \left. - \frac{\sigma}{\sigma+2c} S_t^{1-\kappa} M_t^\kappa N\left(-\frac{b_0+\tau c}{\sqrt{\tau}}\right) + \frac{2c+\sigma+\sigma}{\sigma+2c} e^{r\tau} S_t N\left(-\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\right) \right\} - S_t \\ &= e^{-r\tau} \left\{ M_t N\left(\frac{b_0-\tau c}{\sqrt{\tau}}\right) - \frac{\sigma}{\sigma+2c} S_t^{1-\kappa} M_t^\kappa N\left(-\frac{b_0+\tau c}{\sqrt{\tau}}\right) \right. \\ &\quad \left. + e^{r\tau} S_t N\left(-\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\right) + \frac{\sigma}{\sigma+2c} e^{r\tau} S_t N\left(-\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\right) \right\} - S_t \quad (11.28) \end{aligned}$$

We have

$$\begin{aligned} -\frac{b_0+\tau c}{\sqrt{\tau}} &= -\frac{\sigma b_0 + \sigma c\tau}{\sigma\sqrt{\tau}} = -\frac{\log\left[\frac{M_t}{S_t}\right] + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\log\left[\frac{S_t}{M_t}\right] - (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} =: d_0 - \frac{r}{\sigma}\sqrt{\tau} =: d_{0,-} \\ -\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}} &= -\frac{\sigma b_0 - \sigma(c+\sigma)\tau}{\sigma\sqrt{\tau}} = -\frac{\log\left[\frac{M_t}{S_t}\right] - (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\log\left[\frac{S_t}{M_t}\right] + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_0 + \frac{r}{\sigma}\sqrt{\tau} =: d_{0,+} \end{aligned}$$

with

$$d_0 := \frac{\log\left[\frac{S_t}{M_t}\right] + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$$

and

$$\frac{b_0-\tau c}{\sqrt{\tau}} = \frac{\sigma b_0 - \sigma c\tau}{\sigma\sqrt{\tau}} = \frac{\log\left[\frac{M_t}{S_t}\right] - (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} =: d_1$$

Using this, (18.24) can be written as

$$\begin{aligned} V_t &= e^{-r\tau} \left\{ M_t N\left(\frac{b_0-\tau c}{\sqrt{\tau}}\right) + e^{r\tau} S_t N\left(-\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\right) \right. \\ &\quad \left. + \frac{\sigma}{\sigma+2c} e^{r\tau} S_t N\left(-\frac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\right) - \frac{\sigma}{\sigma+2c} S_t^{1-\kappa} M_t^\kappa N\left(-\frac{b_0+\tau c}{\sqrt{\tau}}\right) \right\} - S_t \\ &= e^{-r\tau} \left\{ M_t N(d_1) + e^{r\tau} S_t N(d_{0,+}) \right. \\ &\quad \left. + \frac{1}{\kappa} \left[e^{r\tau} S_t N(d_{0,+}) - S_t^{1-\kappa} M_t^\kappa N(d_{0,-}) \right] \right\} - S_t \end{aligned}$$

and the theorem is proven. ■