Chapter 19

The Black-Scholes-Vasicek Model

The Black-Scholes-Vasicek model is given by a standard time-dependent Black-Scholes model for the stock price process S_t, with time-dependent but deterministic volatility $\sigma_{S,t}$, and with interest rates $r = r_t$ which are assumed to be not constant, but stochastic. That is, under the risk neutral measure, (we ignore dividends)

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma_{S,t}\, dB_t^S \tag{19.1}$$

with interest rates given by a mean reverting Ornstein-Uhlenbeck or Vasicek process,

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r dB_t^r \tag{19.2}$$

with a constant interest rate volatility σ_r. This model is in particular relevant for pricing longer dated options with maturities of 5, 10 or 20 years. The Brownian motions dB^S and dB^r may have a non vanishing correlation,

$$dB_t^S \cdot dB_t^r = \rho\, dt \tag{19.3}$$

The solutions to (19.1) and (19.2) are given by $(t > s)$

$$S_t = S_s e^{\int_s^t (r_u - \sigma_{S,u}^2/2)du + \int_s^t \sigma_{S,u} dB_u^S} \tag{19.4}$$

and

$$r_t = e^{-\kappa(t-s)} r_s + \bar{r}(1 - e^{-\kappa(t-s)}) + \sigma_r e^{-\kappa t} \int_s^t e^{\kappa u} dB_u^r \tag{19.5}$$

For the model (19.1,19.2), there is a closed form solution for plain vanilla calls and puts. For a constant (time independent) stock volatility $\sigma_{S,t} = \sigma_S$, a derivation can be found in [1]. For time dependent volatility the derivation is similar. There is the following

Theorem 19.1: Let, under the risk neutral measure, the stock and interest rate processes (S_t, r_t) be given by (19.1,19.2) and let $g_\kappa(t)$ be given by

$$g_\kappa(t) \;:=\; \tfrac{1-e^{-\kappa t}}{\kappa} \tag{19.6}$$

Then the following statements hold:

a) Let $H = H(S_T, r_T)$ be some non exotic payoff with maturity T. Then:

$$\begin{aligned} V(S_t, r_t, t, T) \;&:=\; \mathsf{E}\left[e^{-\int_t^T r_s ds} H(S_T, r_T)\,|\, S_t, r_t\right] \\ &= \; P(t,T)\int_{\mathbb{R}^2} H\Big(\tfrac{S_t}{P(t,T)}\, e^{-\frac{v_{t,T}}{2}+\sqrt{v_{t,T}}\,\phi}, r_{t,T}(\phi,\xi)\Big) e^{-\frac{\phi^2+\xi^2}{2}}\,\tfrac{d\phi\, d\xi}{2\pi} \end{aligned} \tag{19.7}$$

where

$$\begin{aligned} r_{t,T}(\phi,\xi) \;&:=\; \bar{r} + e^{-\kappa(T-t)}(r_t - \bar{r}) - \tfrac{\sigma_r^2}{2}\, g_\kappa(T-t)^2 \\ &\quad + \sigma_r\sqrt{g_{2\kappa}(T-t)} \times \left[\rho_{t,T}\,\phi + (1-\rho_{t,T}^2)^{1/2}\,\xi\right] \end{aligned} \tag{19.8}$$

$$\rho_{t,T} \;:=\; \frac{\rho\int_t^T \sigma_{S,u}\, e^{-\kappa(T-u)} du \;+\; \frac{\sigma_r}{2}\, g_\kappa(T-t)^2}{\sqrt{v_{t,T}}\sqrt{g_{2\kappa}(T-t)}} \tag{19.9}$$

$v_{t,T}$ is given in (b) below and $P(t,T)$ is the price of a zero bond in the Vasicek model,

$$\begin{aligned} P(t,T) \;&:=\; \mathsf{E}\left[e^{-\int_t^T r_s ds}\,|\, r_t\right] \\ &= \; A(t,T)\, e^{-g_\kappa(T-t)\, r_t} \end{aligned} \tag{19.10}$$

$$A(t,T) \;=\; e^{\left(\bar{r}-\frac{\sigma_r^2}{2\kappa^2}\right)\{g_\kappa(T-t)\,-\,[T-t]\}\;-\;\frac{\sigma_r^2}{4\kappa}\, g_\kappa(T-t)^2} \tag{19.11}$$

b) The price of a plain vanilla european call/put is given by ($\epsilon = +1$ for call, $\epsilon = -1$ for put)

$$V(S_t, t, T) \;=\; \epsilon\left\{S_t\, N(\epsilon d_+) - P(t,T)\, K N(\epsilon d_-)\right\} \tag{19.12}$$

where

$$d_\pm \;=\; \frac{\log\left[\frac{S_t/K}{P(t,T)}\right] \;\pm\; v_{t,T}/2}{\sqrt{v_{t,T}}} \tag{19.13}$$

$$v_{t,T} \;=\; \int_t^T \Big\{\sigma_{S,u}^2 + 2\,\rho\,\sigma_r\,\sigma_{S,u}\, g_\kappa(T-u) + \sigma_r^2\, g_\kappa(T-u)^2\Big\} du \tag{19.14}$$

Remarks: (i) Since interest rates have dimension $1/time$, the IR volatility σ_r has dimension $1/time^{\frac{3}{2}}$. The stock volatility has dimension $1/\sqrt{time}$ and κ has dimension $1/time$. Thus $v_{t,T}$ in (19.14) above is indeed a dimensionless quantity as it should be.

(ii) For constant stock volatility $\sigma_{S,u} = \sigma_S =: \sigma$,

$$v_{t,T} = \sigma^2(T-t) + 2\rho\,\sigma_r\,\sigma \int_t^T \tfrac{1-e^{-\kappa(T-u)}}{\kappa}\,du + \tfrac{\sigma_r^2}{\kappa^2}\Big(T-t-2\tfrac{1-e^{-\kappa(T-t)}}{\kappa} + \tfrac{1-e^{-2\kappa(T-t)}}{2\kappa}\Big)$$

$$= \sigma^2\,(T-t) + 2\rho\,\sigma\,\sigma_r\,\tfrac{1}{\kappa}\Big(T-t-\tfrac{1-e^{-\kappa(T-t)}}{\kappa}\Big) + \tfrac{\sigma_r^2}{\kappa^2}\Big(T-t-2\tfrac{1-e^{-\kappa(T-t)}}{\kappa} + \tfrac{1-e^{-2\kappa(T-t)}}{2\kappa}\Big)$$

or, with $\tau := T-t$,

$$v \;=\; v(\tau) \;=\; \sigma^2 \times \tau \;+\; 2\rho\,\sigma\,\sigma_r \times \tfrac{\tau - g_\kappa(\tau)}{\kappa} \;+\; \sigma_r^2 \times \tfrac{\tau - 2g_\kappa(\tau) + g_{2\kappa}(\tau)}{\kappa^2} \tag{19.15}$$

and

$$\rho_{t,T} \;=\; \rho(\tau) \;=\; \frac{\rho\,\sigma\,g_\kappa(\tau) \;+\; \frac{\sigma_r}{2} g_\kappa(\tau)^2}{\sqrt{v(\tau)}\sqrt{g_{2\kappa}(\tau)}} \tag{19.16}$$

Proof of Theorem: We have to compute the expectation

$$\begin{aligned} V(S_t, r_t, t, T) \;&=\; \mathsf{E}\left[e^{-\int_t^T r_s ds} H(S_T, r_T)\,|\,S_t, r_t\right] \\ &=\; \mathsf{E}\left[e^{-\int_t^T r_s ds} H\Big(S_t\, e^{\int_t^T (r_u - \sigma_{S,u}^2/2)du} e^{\int_t^T \sigma_{S,u} dB_u^S}, r_T\Big)\,|\,S_t, r_t\right] \end{aligned} \tag{19.17}$$

with

$$S_T \;=\; S_t e^{\int_t^T (r_u - \sigma_{S,u}^2/2)du + \int_t^T \sigma_{S,u} dB_u^S} \tag{19.18}$$

$$r_T \;=\; e^{-\kappa(T-t)} r_t + \bar{r}(1 - e^{-\kappa(T-t)}) + \sigma_r\, e^{-\kappa T} \int_t^T e^{\kappa u} dB_u^r \tag{19.19}$$

satisfy the SDE's

$$\tfrac{dS_t}{S_t} \;=\; r_t\,dt + \sigma_{S,t}\,dB_t^S \tag{19.20}$$

$$dr_t \;=\; \kappa(\bar{r} - r_t)dt + \sigma_r dB_t^r \tag{19.21}$$

with correlated Brownian motions

$$dB_t^S \cdot dB_t^r = \rho\,dt \tag{19.22}$$

We have

$$\begin{aligned} \int_t^T r_u du \;&=\; \tfrac{1-e^{-\kappa(T-t)}}{\kappa}\,r_t + \bar{r}\Big(T-t-\tfrac{1-e^{-\kappa(T-t)}}{\kappa}\Big) + \sigma_r \int_t^T \tfrac{1-e^{-\kappa(T-v)}}{\kappa}\,dB_v^r \\ &=:\; \mu_{t,T} \;+\; \sigma_r \int_t^T g_\kappa(T-v)\,dB_v^r \end{aligned} \tag{19.23}$$

where we introduced the notation

$$\mu_{t,T} = \tfrac{1-e^{-\kappa(T-t)}}{\kappa} r_t + \bar{r}\left(T - t - \tfrac{1-e^{-\kappa(T-t)}}{\kappa}\right) \tag{19.24}$$

$$g_\kappa(t) = \tfrac{1-e^{-\kappa t}}{\kappa} \tag{19.25}$$

Introducing further

$$\eta_{t,T} = \tfrac{1}{2}\int_t^T \sigma_{S,u}^2 \, du \tag{19.26}$$

and

$$dB_t^r \equiv dB_t \tag{19.27}$$

$$dB_t^S = \rho \, dB_t + \sqrt{1-\rho^2}\, dZ_t \tag{19.28}$$

with two independent Brownian motions dB_t and dZ_t, the expectation (19.17) reads

$$\begin{aligned} V(S_t, r_t, t, T) \;=\; & e^{-\mu_{t,T}} \int e^{-\sigma_r \int_t^T g_\kappa(T-u)\, dB_u} \times \\ & H\Big(S_t \, e^{\mu_{t,T} - \eta_{t,T} + \sigma_r \int_t^T g_\kappa(T-u) dB_u + \int_t^T \sigma_{S,u}(\rho dB_u + \sqrt{1-\rho^2} dZ_u)}, r_T\Big) dW(B) dW(Z) \end{aligned} \tag{19.29}$$

where $dW(B) = dW(\{B_u\}_{t<u\le T})$ denotes the Wiener measure with respect to B.

To eliminate the factor $e^{-\sigma_r \int_t^T g_\kappa(T-u)\, dB_u}$ in the first line of (19.29), we make the substitution of variables $d\tilde{B}_u = dB_u + \sigma_r g_\kappa(T-u) du$ or, for $s \in (t, T]$,

$$\tilde{B}_s = B_s + \sigma_r \int_t^s g_\kappa(T-u)\, du \tag{19.30}$$

and leave Z unchanged. Then, by Girsanov's Theorem

$$e^{-\sigma_r \int_t^T g_\kappa(T-u)\, dB_u - \frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} dW(B) = dW(\tilde{B}) \tag{19.31}$$

such that

$$\begin{aligned} V(S_t, r_t, t, T) \;=\; & e^{-\mu_{t,T}} e^{\frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \times \\ & \int H\Big(S_t \, e^{\mu_{t,T} - \eta_{t,T} + \sigma_r \int_t^T g_\kappa(T-u)[d\tilde{B}_u - \sigma_r g_\kappa(T-u)\, du]} \times \\ & \qquad e^{\int_t^T \sigma_{S,u}\left\{\rho[d\tilde{B}_u - \sigma_r g_\kappa(T-u)\, du] + \sqrt{1-\rho^2} dZ_u\right\}}, r_T\Big) dW(\tilde{B}) dW(Z) \\ =\; & e^{-\mu_{t,T} + \frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \times \\ & \int H\Big(S_t \, e^{\mu_{t,T} - \eta_{t,T} - \sigma_r^2 \int_t^T g_\kappa(T-u)^2 du - \rho\sigma_r \int_t^T \sigma_{S,u} g_\kappa(T-u)\, du} \times \\ & \qquad e^{\int_t^T \left\{\rho\sigma_{S,u} + \sigma_r g_\kappa(T-u)\right\} d\tilde{B}_u + \sqrt{1-\rho^2}\int_t^T \sigma_{S,u} dZ_u}, r_T\Big) dW(\tilde{B}) dW(Z) \end{aligned} \tag{19.32}$$

with r_T given by

$$\begin{aligned}
r_T &= e^{-\kappa(T-t)}r_t + \bar{r}(1-e^{-\kappa(T-t)}) + \sigma_r \int_t^T e^{-\kappa(T-u)} dB_u^r \\
&= e^{-\kappa(T-t)}r_t + \bar{r}(1-e^{-\kappa(T-t)}) + \sigma_r \int_t^T e^{-\kappa(T-u)}\big(d\tilde{B}_u - \sigma_r g_\kappa(T-u)du\big) \\
&= e^{-\kappa(T-t)}r_t + \bar{r}(1-e^{-\kappa(T-t)}) - \tfrac{\sigma_r^2}{2} g_\kappa(T-t)^2 + \sigma_r \int_t^T e^{-\kappa(T-u)} d\tilde{B}_u \\
&=: q_{t,T} + Y_{t,T} \qquad (19.33)
\end{aligned}$$

where we abbreviated

$$q_{t,T} = e^{-\kappa(T-t)}r_t + \bar{r}(1-e^{-\kappa(T-t)}) - \tfrac{\sigma_r^2}{2} g_\kappa(T-t)^2 \qquad (19.34)$$

$$Y_{t,T} = \sigma_r \int_t^T e^{-\kappa(T-u)} d\tilde{B}_u \qquad (19.35)$$

and we used the relation

$$\begin{aligned}
\int_t^T e^{-\kappa(T-u)} g_\kappa(T-u)du &= \int_t^T \tfrac{e^{-\kappa(T-u)} - e^{-2\kappa(T-u)}}{\kappa} du \\
&= \tfrac{1}{\kappa^2}\big(1 - e^{-\kappa(T-t)} - \tfrac{1-e^{-2\kappa(T-t)}}{2}\big) \\
&= \tfrac{1}{2\kappa^2}\big(1 - 2e^{-\kappa(T-t)} + e^{-2\kappa(T-t)}\big) \\
&= \tfrac{1}{2}\, g_\kappa(T-t)^2 \qquad (19.36)
\end{aligned}$$

The random variables

$$X_{t,T} := \int_t^T \{\rho\sigma_{S,u} + \sigma_r g_\kappa(T-u)\}\, d\tilde{B}_u + \sqrt{1-\rho^2} \int_t^T \sigma_{S,u} dZ_u \qquad (19.37)$$

$$Y_{t,T} = \sigma_r \int_t^T e^{-\kappa(T-u)}\, d\tilde{B}_u \qquad (19.38)$$

are Gaussian with mean 0 and variances

$$\begin{aligned}
v_{t,T} := \mathsf{V}[X_{t,T}] &= \int_t^T \Big\{\big[\rho\sigma_{S,u} + \sigma_r g_\kappa(T-u)\big]^2 + (1-\rho^2)\sigma_{S,u}^2\Big\}\, du \qquad (19.39) \\
w_{t,T} := \mathsf{V}[Y_{t,T}] &= \sigma_r^2 \int_t^T e^{-2\kappa(T-u)} du \\
&= \sigma_r^2\, g_{2\kappa}(T-t) \qquad (19.40) \\
\mathsf{Cov}(X_{t,T}, Y_{t,T}) &= \int_t^T \big\{\rho\sigma_r\sigma_{S,u}\, e^{-\kappa(T-u)} + \sigma_r^2 g_\kappa(T-u) e^{-\kappa(T-u)}\big\} du \\
&= \rho\sigma_r \int_t^T \sigma_{S,u}\, e^{-\kappa(T-u)} du + \tfrac{\sigma_r^2}{2} g_\kappa(T-t)^2 \qquad (19.41)
\end{aligned}$$

where we used (19.36) again in the last line. Thus, introducing

$$F_{t,T} := \mu_{t,T} - \eta_{t,T} - \sigma_r^2 \int_t^T g_\kappa(T-u)^2 du - \rho\sigma_r \int_t^T \sigma_{S,u} g_\kappa(T-u)\, du \qquad (19.42)$$

and recalling $r_T = q_{t,T} + Y_{t,T}$, we obtain

$$\begin{aligned} V(S_t, r_t, t, T) &= e^{-\mu_{t,T}+\frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \times \\ & \int_{\mathbb{R}^2} H\big(S_t\, e^{F_{t,T}+X_{t,T}}, q_{t,T} + Y_{t,T}\big) e^{-\frac{1}{2}(X_{t,T},Y_{t,T})C_{t,T}^{-1}\binom{X_{t,T}}{Y_{t,T}}} \det C_{t,T} \frac{dX_{t,T}dY_{t,T}}{2\pi} \\ &= e^{-\mu_{t,T}+\frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \times \qquad (19.43) \\ & \int_{\mathbb{R}^2} H\big(S_t\, e^{F_{t,T}+\sqrt{v_{t,T}}\,x}, q_{t,T} + \sqrt{w_{t,T}}\,y\big) e^{-\frac{1}{2}(x,y)R_{t,T}^{-1}\binom{x}{y}} \det R_{t,T} \frac{dx\,dy}{2\pi} \qquad (19.44) \end{aligned}$$

where

$$C_{t,T} = \begin{pmatrix} \mathsf{V}[X_{t,T}] & \mathsf{Cov}(X_{t,T}, Y_{t,T}) \\ \mathsf{Cov}(X_{t,T}, Y_{t,T}) & \mathsf{V}[Y_{t,T}] \end{pmatrix} \qquad (19.45)$$

denotes the covariance matrix of $X_{t,T}$ and $Y_{t,T}$ and $R_{t,T}$ the corresponding correlation matrix,

$$R_{t,T} = \begin{pmatrix} 1 & \rho_{t,T} \\ \rho_{t,T} & 1 \end{pmatrix} \qquad (19.46)$$

with

$$\begin{aligned} \rho_{t,T} &= \frac{\mathsf{Cov}(X_{t,T}, Y_{t,T})}{\sqrt{\mathsf{V}[X_{t,T}]\mathsf{V}[Y_{t,T}]}} \\ &= \frac{\rho\sigma_r \int_t^T \sigma_{S,u}\, e^{-\kappa(T-u)} du + \frac{\sigma_r^2}{2} g_\kappa(T-t)^2}{\sqrt{v_{t,T}}\sqrt{\sigma_r^2\, g_{2\kappa}(T-t)}} \\ &= \frac{\rho \int_t^T \sigma_{S,u}\, e^{-\kappa(T-u)} du + \frac{\sigma_r}{2} g_\kappa(T-t)^2}{\sqrt{v_{t,T}}\sqrt{g_{2\kappa}(T-t)}} \qquad (19.47) \end{aligned}$$

Transforming to independent Gaussians,

$$\begin{aligned} V &= e^{-\mu_{t,T}+\frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \times \qquad (19.48) \\ & \int_{\mathbb{R}^2} H\Big(S_t\, e^{F_{t,T}+\sqrt{v_{t,T}}\,x}, q_{t,T} + \sqrt{w_{t,T}}\big(\rho_{t,T} x + (1-\rho_{t,T}^2)^{1/2}\, y\big)\Big) e^{-\frac{x^2+y^2}{2}} \frac{dx\,dy}{2\pi} \end{aligned}$$

For the special case $H \equiv 1$, (19.17) and the last equation reduce to

$$\begin{aligned} V = P(t,T) &= \mathsf{E}\big[e^{-\int_t^T r_u du} \mid r_t\big] \\ &= e^{-\mu_{t,T}+\frac{1}{2}\sigma_r^2 \int_t^T g_\kappa(T-u)^2 du} \qquad (19.49) \end{aligned}$$

With that, we obtain

$$\begin{aligned} e^{F_{t,T}} &= e^{\mu_{t,T}-\sigma_r^2\int_t^T g_\kappa(T-u)^2du-\rho\sigma_r\int_t^T \sigma_{S,u}g_\kappa(T-u)\,du-\eta_{t,T}} \\ &= e^{-\frac{1}{2}\sigma_r^2\int_t^T g_\kappa(T-u)^2du-\rho\sigma_r\int_t^T \sigma_{S,u}g_\kappa(T-u)\,du-\eta_{t,T}}\Big/P(t,T) \\ &=: e^{m_{t,T}}/P(t,T) \end{aligned} \tag{19.50}$$

where we introduced

$$m_{t,T} := -\tfrac{1}{2}\sigma_r^2\int_t^T g_\kappa(T-u)^2du-\rho\sigma_r\int_t^T \sigma_{S,u}g_\kappa(T-u)\,du-\eta_{t,T} \tag{19.51}$$

From the definition of $v_{t,T}$ in (19.39)

$$\begin{aligned} v_{t,T} &= \int_t^T \Big\{\big[\rho\sigma_{S,u}+\sigma_r g_\kappa(T-u)\big]^2+(1-\rho^2)\sigma_{S,u}^2\Big\}\,du \\ &= \int_t^T \left\{\sigma_{S,u}^2+2\rho\sigma_r\sigma_{S,u}g_\kappa(T-u)+\sigma_r^2 g_\kappa(T-u)^2\right\}du \end{aligned} \tag{19.52}$$

such that, recalling (19.26),

$$m_{t,T} = -\tfrac{v_{t,T}}{2} \tag{19.53}$$

Thus, now for a general H again, continuing with (19.48),

$$\begin{aligned} V(S_t,t,T) &= P(t,T)\int_{\mathbb{R}^2} H\Big(\tfrac{S_t}{P(t,T)}\,e^{m_{t,T}+\sqrt{v_{t,T}}\,x},x_{t,T}(x,y)\Big)e^{-\frac{x^2+y^2}{2}}\,\tfrac{dx\,dy}{2\pi} \\ &= P(t,T)\int_{\mathbb{R}^2} H\Big(\tfrac{S_t}{P(t,T)}\,e^{-\frac{v_{t,T}}{2}+\sqrt{v_{t,T}}\,x},x_{t,T}(x,y)\Big)e^{-\frac{x^2+y^2}{2}}\,\tfrac{dx\,dy}{2\pi} \end{aligned} \tag{19.54}$$

where

$$\begin{aligned} x_{t,T}(x,y) &= q_{t,T}+\sqrt{w_{t,T}}\big(\rho_{t,T}x+(1-\rho_{t,T}^2)^{1/2}\,y\big) \\ &= \bar{r}+e^{-\kappa(T-t)}(r_t-\bar{r})-\tfrac{\sigma_r^2}{2}g_\kappa(T-t)^2 \\ &\quad +\sigma_r\sqrt{g_{2\kappa}(T-t)}\times\big[\rho_{t,T}\,x+(1-\rho_{t,T}^2)^{1/2}\,y\big] \end{aligned} \tag{19.55}$$

In particular, for a plain vanilla call $H(S_T)=(S_T-K)_+$, the same computation as in the Black-Scholes case (see chapter 6) gives

$$\begin{aligned} V_{\text{Call}}(S_t,r_t,t,T) &= P(t,T)\,\Big\{\tfrac{S_t}{P(t,T)}N(d_+)-KN(d_-)\Big\} \\ &= S_t\,N(d_+)-P(t,T)\,KN(d_-) \end{aligned} \tag{19.56}$$

where

$$d_\pm = \frac{\log\Big[\frac{S_t/K}{P(t,T)}\Big]\pm\frac{v_{t,T}}{2}}{\sqrt{v_{t,T}}} \tag{19.57}$$

and this proves the theorem. ■

Calibration of the Model to European Call Options

We assume that the interest rate process has already been calibrated to the initial yield curve (by using a suitable shift function $\int_0^T r_t dt \to \int_0^T \{r_t + \varphi(t)\}dt$ with a piecewise constant $\varphi(t)$) as well as to swaptions and caplets. Now we want to fit the model to ATM call options on S with the maturities $T_1, T_2, ..., T_n$. Let $\sigma_{\text{imp}}(T_i)$ be the implied volatilities of those call options. We assume that the instantaneous stock volatility

$$\sigma_{S,t} \quad =: \quad \sigma(t)$$

is piecewise constant on the intervalls $[0, T_1)$, $[T_1, T_2)$, ... ,$[T_{n-1}, T_n]$. From the Theorem 19.1 above we find that the implied volatility $\sigma_{\text{BSVas,imp}}$ (squared, times maturity) of a plain vanilla call in the BS-Vasicek model with maturity T is given by

$$\begin{aligned} v_{0,T} \;\equiv\; v_T \;&=\; \sigma^2_{\text{BSVas,imp}}(T) \times T \qquad (19.58) \\ &=\; \int_0^T \left\{\sigma(u)^2 + 2\rho\sigma_r\sigma(u)g_\kappa(T-u) + \sigma_r^2 g_\kappa(T-u)^2\right\}du \end{aligned}$$

For piecewise constant

$$\sigma(u)|_{u\in[T_{i-1},T_i)} \quad =: \quad \sigma_i$$

the above equation becomes (we write temporarily σ^r instead of σ_r for the interest rate vol, to avoid confusion with the equity σ_i's)

$$\begin{aligned} v_{T_i} \;&=\; \sum_{j=1}^{i} \int_{T_{j-1}}^{T_j} \{\sigma(u)^2 + 2\rho\sigma^r\sigma(u)g_\kappa(T_i-u) + (\sigma^r)^2 g_\kappa(T_i-u)^2\}\,du \qquad (19.59) \\ &=\; \sum_{j=1}^{i} \left\{\sigma_j^2(T_j - T_{j-1}) + 2\rho\sigma^r\sigma_j \int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)du + (\sigma^r)^2 \int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)^2du\right\} \end{aligned}$$

with $T_0 := 0$. Now the calibration is done by requiring

$$\sigma_{\text{BSVas,imp}}(T_i) \;=\; \sigma^{\text{model}}_{\text{imp}}(T_i) \;\stackrel{!}{=}\; \sigma^{\text{market}}_{\text{imp}}(T_i) \;\equiv\; \sigma_{\text{imp}}(T_i) \qquad (19.60)$$

for all $i = 1, 2, ..., n$. The right hand side of (19.60) is taken from market data, and the left hand side is given by the expression (19.59). Thus we are led to the following system of quadratic equations for σ_i: For all $i = 1, 2, ..., n$

$$\sigma^2_{\text{imp}}(T_i) \;=\; \sum_{j=1}^{i} \left\{\sigma_j^2 + 2\sigma_j \tfrac{\rho\sigma^r}{\Delta T_j} \int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)du + \tfrac{(\sigma^r)^2}{\Delta T_j} \int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)^2du\right\} \tfrac{\Delta T_j}{T_i} \qquad (19.61)$$

where we put $\Delta T_j = T_j - T_{j-1}$. From this set, $\sigma_1, \sigma_2, ..., \sigma_n$ can be determined inductively starting with σ_1: For $i = 1$, the above equation simplifies to

$$\sigma^2_{\rm imp}(T_1) = \Big\{\sigma_1^2 + 2\sigma_1 \tfrac{\rho\sigma^r}{\Delta T_1} \int_{T_0}^{T_1} g_\kappa(T_1-u)du + \tfrac{(\sigma^r)^2}{\Delta T_1} \int_{T_0}^{T_1} g_\kappa(T_1-u)^2 du\Big\} \quad (19.62)$$

which gives

$$\begin{aligned}\sigma_1 &= -\rho \tfrac{\sigma^r}{\Delta T_1} \int_{T_0}^{T_1} g_\kappa(T_1-u)du \\ &\quad \pm \sqrt{\sigma^2_{\rm imp}(T_1) + \rho^2 \big(\tfrac{\sigma^r}{\Delta T_1} \int_{T_0}^{T_1} g_\kappa(T_1-u)du\big)^2 - \tfrac{(\sigma^r)^2}{\Delta T_1} \int_{T_0}^{T_1} g_\kappa(T_1-u)^2 du}\end{aligned} \quad (19.63)$$

We have

$$\tfrac{1}{t-s}\int_s^t g_\kappa(t-u)du = \tfrac{1}{\kappa}\big(1 - \tfrac{1-e^{-\kappa(t-s)}}{\kappa(t-s)}\big) \quad (19.64)$$

$$\tfrac{1}{t-s}\int_s^t g_\kappa(t-u)^2du = \tfrac{1}{\kappa^2}\big(1 - 2\tfrac{1-e^{-\kappa(t-s)}}{\kappa(t-s)} + \tfrac{1-e^{-2\kappa(t-s)}}{2\kappa(t-s)}\big) \quad (19.65)$$

For small $x := \kappa(t-s) < 1$ we may Taylor expand to obtain

$$\begin{aligned}\tfrac{1-e^{-x}}{x} &= \tfrac{1-(1-x+x^2/2-x^3/6+O(x^4))}{x} \\ &= 1 - \tfrac{x}{2} + \tfrac{x^2}{6} + O(x^3)\end{aligned} \quad (19.66)$$

Thus,

$$\begin{aligned}\tfrac{1}{t-s}\int_s^t g_\kappa(t-u)du &= \tfrac{1}{\kappa}\Big(1 - \big(1 - \tfrac{x}{2} + \tfrac{x^2}{6} + O(x^3)\big)\Big) \\ &= \tfrac{1}{\kappa}\big(\tfrac{x}{2} + \tfrac{x^2}{6} + O(x^3)\big) \\ &= \tfrac{t-s}{2} + O\big(\kappa(t-s)^2\big)\end{aligned} \quad (19.67)$$

$$\begin{aligned}\tfrac{1}{t-s}\int_s^t g_\kappa(t-u)^2du &= \tfrac{1}{\kappa^2}\Big(1 - 2\big[1 - \tfrac{x}{2} + \tfrac{x^2}{6}\big] + 1 - x + \tfrac{2x^2}{3} + O(x^3)\Big) \\ &= \tfrac{1}{\kappa^2}\big(\tfrac{x^2}{3} + O(x^3)\big) \\ &= \tfrac{(t-s)^2}{3} + O\big(\kappa(t-s)^3\big)\end{aligned} \quad (19.68)$$

Realistic data from IR calibration are κ around 10 percent and σ^r around 0.5 to 1 percent, (at least in pre-financial-crisis times)

$$\kappa \approx 10\%, \qquad \sigma^r \approx 1\% \quad (19.69)$$

Then, since $\sigma_{\rm imp}(T)$ is around 20 to 40 percent, for maturities less than, say, 5 years such that $\kappa T_i \leq 0.5$ to justify a Taylor expansion,

$$\begin{aligned}\sigma^2_{\rm imp}(T_1) - \tfrac{(\sigma^r)^2}{\Delta T_1}\int_{T_0}^{T_1} g_\kappa(T_1-u)^2du &\sim \sigma^2_{\rm imp}(T_1) - (\sigma^r)^2 T_1^2/3 \\ &\sim \sigma^2_{\rm imp}(T_1) - \tfrac{(T_1/100)^2}{3}\end{aligned} \quad (19.70)$$

should be positive. If this is not the case $\sigma_1 = 0$ may be a reasonable choice. Now suppose that $\sigma_1, ..., \sigma_{i-1}$ have already been determined. Then σ_i is obtained from

$$\begin{aligned} \sigma^2_{\text{imp}}(T_i) &= \Big\{ \sigma_i^2 + 2\sigma_i \tfrac{\rho\sigma^r}{\Delta T_i} \textstyle\int_{T_{i-1}}^{T_i} g_\kappa(T_i-u)du + \tfrac{(\sigma^r)^2}{\Delta T_i} \textstyle\int_{T_{i-1}}^{T_i} g_\kappa(T_i-u)^2 du \Big\} \tfrac{\Delta T_i}{T_i} \\ &\quad + s_{i-1}(\sigma_1, ..., \sigma_{i-1}) \end{aligned} \quad (19.71)$$

where we put

$$s_{i-1} = \sum_{j=1}^{i-1} \Big\{ \sigma_j^2 + 2\sigma_j \tfrac{\rho\sigma^r}{\Delta T_j} \textstyle\int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)du + \tfrac{(\sigma^r)^2}{\Delta T_j} \textstyle\int_{T_{j-1}}^{T_j} g_\kappa(T_i-u)^2 du \Big\} \tfrac{\Delta T_j}{T_i} \quad (19.72)$$

If one would substitute in the above expression T_i by T_{i-1}, one would obtain the expression for $\sigma^2_{\text{imp}}(T_{i-1})$. Therefore we abreviate

$$\Delta\sigma^2_{\text{imp},i} := \sigma^2_{\text{imp}}(T_i) - s_{i-1}(\sigma_1, ..., \sigma_{i-1}) \quad (19.73)$$

Then,

$$\begin{aligned} \sigma_i &= -\rho \tfrac{\sigma^r}{\Delta T_i} \textstyle\int_{T_{i-1}}^{T_i} g_\kappa(T_i-u)du \\ &\quad + \sqrt{T_i \tfrac{\Delta\sigma^2_{\text{imp},i}}{\Delta T_i} + \rho^2 \big(\tfrac{\sigma^r}{\Delta T_i} \textstyle\int_{T_{i-1}}^{T_i} g_\kappa(T_i-u)du\big)^2 - \tfrac{(\sigma^r)^2}{\Delta T_i} \textstyle\int_{T_{i-1}}^{T_i} g_\kappa(T_i-u)^2 du} \end{aligned} \quad (19.74)$$

if the above expression is a positive real number and $\sigma_i = 0$ otherwise. Finally,

$$\begin{aligned} \tfrac{1}{t-s} \textstyle\int_s^t g_\kappa(T-u)du &= \tfrac{1}{\kappa}\big(1 - \tfrac{e^{-\kappa(T-t)} - e^{-\kappa(T-s)}}{\kappa(t-s)}\big) \\ \tfrac{1}{t-s} \textstyle\int_s^t g_\kappa(T-u)^2 du &= \tfrac{1}{\kappa^2}\big(1 - 2\tfrac{e^{-\kappa(T-t)} - e^{-\kappa(T-s)}}{\kappa(t-s)} + \tfrac{e^{-2\kappa(T-t)} - e^{-2\kappa(T-s)}}{2\kappa(t-s)}\big) \end{aligned}$$