Chapter 20

Bessel Processes

We follow the discussion of section one of chapter XI of [17]. Let $B_t = (B^1, ..., B^n)$ be an uncorrelated d-dimensional Brownian motion. Then the process

$$\rho_t \;=\; |B_t| \;=\; \sqrt{(B_t^1)^2 + \cdots + (B_t^n)^2} \tag{20.1}$$

is called an n-dimensional Bessel process and $Z_t = \rho_t^2$ is called a squared Bessel process. Using Ito's formula,

$$dZ_t \;=\; 2\sum_{i=1}^{n} B_t^i \, dB_t^i \;+\; n\, dt \tag{20.2}$$

Consider the process

$$W_t \;:=\; \sum_{i=1}^{n} \int_0^t \frac{B_s^i}{\rho_s} \, dB_s^i \tag{20.3}$$

For $n > 1$, $\rho_t > 0$ almost surely and for $d = 1$ the set $\{s \,|\, \rho_s = 0\}$ has Lebesgue measure zero almost surely such that W_t is well defined. Because of the quadratic variation

$$\langle W_t, W_t \rangle \;=\; \sum_{i=1}^{n} \int_0^t \frac{(B_s^i)^2}{\rho_s^2} \, ds \;=\; t \tag{20.4}$$

W_t coincides with a one dimensional Brownian motion. Thus Z_t is the solution of the SDE

$$dZ_t \;=\; 2\sqrt{Z_t}\, dW_t \;+\; n\, dt \tag{20.5}$$

Although the square root is only Hölder continuous, but not Lipshitz, equation (20.5) has a unique solution for all $n \geq 0$ (not necessarily an integer) and $Z_0 = x \geq 0$, see for example [17], section 3 of chapter IX. Thus Z_t is a time homogenous Markov process which is determined by its transition function $q_t(x, y) = q_t^n(x, y) = q^n(0, x; t, y)$ given by

$$q^n(s, x\,;\, t, y)\, dy \;=\; \mathsf{P}\big[Z_t^{s,x;n} \in [y, y + dy)\,\big] \tag{20.6}$$

where $Z_t^{s,x;n}$ denotes the solution of (20.5) with initial condition $Z_s^{s,x;n} = x$. Finally we introduce the probability density

$$q_t^{n,x}(y) \quad := \quad q^n(0,x\,;\,t,y) \tag{20.7}$$

such that

$$\mathsf{E}\big[F(Z_t^{0,x;n})\big] \quad = \quad \int_{\mathbb{R}} F(y)\, q_t^{n,x}(y)\, dy \tag{20.8}$$

There is the following

Theorem 20.1:

a) Let $q_t^{n,x}$ be the probability density for an n-dimensional squared Bessel process starting at $x \in \mathbb{R}^+$ at time zero given by (20.7), $n \in \mathbb{R}^+$. Then for all $n_1, n_2, x_1, x_2 \geq 0$

$$q_t^{n_1,x_1} * q_t^{n_2,x_2} \quad = \quad q_t^{n_1+n_2,x_1+x_2} \tag{20.9}$$

That is,

$$\textstyle\int_{\mathbb{R}} q_t^{n_1,x_1}(z)\, q_t^{n_2,x_2}(y-z)\, dz \quad = \quad q_t^{n_1+n_2,x_1+x_2}(y) \tag{20.10}$$

b) Let μ be some measure on $\mathbb{R}$ and let $Z_t^{0,x;n}$ denotes the solution of (20.5) with initial condition $Z_0^{0,x;n} = x$. Then there exist some numbers A_μ and B_μ such that

$$\mathsf{E}\Big[e^{-\int_0^\infty Z_t^{0,x;n} d\mu(t)}\Big] \quad = \quad A_\mu^x\, B_\mu^n \tag{20.11}$$

c) The transition function (20.6) for the n-dimensional squared Bessel process is given by $q^n(s,x\,;\,t,y) = q_{t-s}^n(x,y)$ where

$$q_t^n(x,y) \quad = \quad \tfrac{1}{2t} \left(\tfrac{y}{x}\right)^{\frac{\nu}{2}} e^{-\frac{x+y}{2t}}\, I_\nu\!\left(\tfrac{\sqrt{xy}}{t}\right), \qquad x,y \geq 0 \tag{20.12}$$

Here $\nu = \frac{n}{2} - 1$ and $I_\nu(z) = \sum_{k=0}^\infty \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}$ is the Bessel function with imaginary argument. In particular, the probability density $q_t^{n,0}$ is given by

$$q_t^{n,0}(y) \quad = \quad \tfrac{1}{2t\,\Gamma(\frac{n}{2})} \left(\tfrac{y}{2t}\right)^{\frac{n}{2}-1} e^{-\frac{y}{2t}}, \qquad y \geq 0 \tag{20.13}$$

Proof: a) We have

$$\begin{aligned}
q_t^{n_1,x_1} * q_t^{n_2,x_2}(y)\, dy &= \textstyle\int_{\mathbb{R}_z} q_t^{n_1,x_1}(z)\, q_t^{n_2,x_2}(y-z)\, dz\, dy \\
&= \textstyle\int_{\mathbb{R}_z} \mathsf{P}\big[Z_t^{0,x_1;n_1} \in [z,z+dz)\big]\, \mathsf{P}\big[Z_t^{0,x_2;n_2} \in [y-z,y-z+dy)\big] \\
&= \textstyle\int_{\mathbb{R}_z} \mathsf{P}\big[Z_t^{0,x_1;n_1} \in [z,z+dz)\big]\, \mathsf{P}\big[z + Z_t^{0,x_2;n_2} \in [y,y+dy)\big] \\
&= \mathsf{P}\big[Z_t^{0,x_1;n_1} + Z_t^{0,x_2;n_2} \in [y,y+dy)\big] \\
&= \mathsf{P}\big[Z_t^{0,x_1+x_2;n_1+n_2} \in [y,y+dy)\big] && (20.14) \\
&= q_t^{n_1+n_2,x_1+x_2}(y)\, dy && (20.15)
\end{aligned}$$

The equality in (20.14) follows from the fact that $Y_t := Z_t^{0,x_1;n_1} + Z_t^{0,x_2;n_2}$ has the same distribution as $Z_t^{0,x_1+x_2;n_1+n_2}$. Namely, Y_t is a solution of the SDE

$$\begin{aligned} dY_t &= 2\Big(\sqrt{Z_t^{0,x_1;n_1}}\,dW_t^1 + \sqrt{Z_t^{0,x_2;n_2}}\,dW_t^2\Big) + (n_1+n_2)\,dt \\ &=: 2\sqrt{Y_t}\,dW_t + (n_1+n_2)\,dt \end{aligned} \tag{20.16}$$

where we defined

$$W_t := \int_0^t \Big(\frac{\sqrt{Z_t^{0,x_1;n_1}}}{\sqrt{Y_t}}\,dW_t^1 + \frac{\sqrt{Z_t^{0,x_2;n_2}}}{\sqrt{Y_t}}\,dW_t^2\Big) \tag{20.17}$$

Since the quadratic variation $\langle W_t, W_t\rangle = \int_0^t \frac{Z_s^{0,x_1;n_1}+Z_s^{0,x_2;n_2}}{Y_s}\,ds = t$, the process W_t is a Brownian motion and part (a) follows.

b) Suppose that $x = x_1 + x_2$ and $n = n_1 + n_2$. Let

$$\phi(x,n) = \mathsf{E}\Big[e^{-\int_0^\infty Z_t^{0,x;n} d\mu(t)}\Big] \tag{20.18}$$

Then, with part (a)

$$\begin{aligned} \phi(x_1+x_2, n_1+n_2) &= \mathsf{E}\Big[e^{-\int_0^\infty (Z_t^{0,x_1;n_1}+Z_t^{0,x_2;n_2}) d\mu(t)}\Big] \\ &= \phi(x_1,n_1)\,\phi(x_2,n_2) \end{aligned} \tag{20.19}$$

which gives

$$\begin{aligned} \phi(x,n) &= \phi(x,0)\,\phi(0,n) \\ &= \phi(1,0)^x\,\phi(0,1)^n \end{aligned} \tag{20.20}$$

This proves part (b).

c) We compute the Laplace transform of $q_t^n(x,y)$,

$$\begin{aligned} \mathsf{E}\big[e^{-\lambda Z_t^{0,x;n}}\big] &= \int_{\mathbb{R}} e^{-\lambda y}\,\mathsf{P}\big[Z_t^{0,x;n} \in [y, y+dy)\big] \\ &= \int_0^\infty e^{-\lambda y}\,q_t^n(x,y)\,dy \end{aligned} \tag{20.21}$$

To obtain the left hand side of (20.21), we put $d\mu(s) = \lambda\delta(s-t)ds$ in part (b) and consider $\phi(x,1) = A_\mu^x\,B_\mu$. Then, if $B_t^{0,\sqrt{x}}$ denotes a one dimensional Brownian motion starting at $\sqrt{x}$ at time zero,

$$\begin{aligned} \phi(x,1) &= \mathsf{E}\Big[e^{-\lambda\int_0^\infty Z_s^{0,x;1}\,\delta(s-t)ds}\Big] = \mathsf{E}\Big[e^{-\lambda Z_t^{0,x;1}}\Big] = \mathsf{E}\Big[e^{-\lambda (B_t^{0,\sqrt{x}})^2}\Big] \\ &= \int_{\mathbb{R}} e^{-\lambda y^2}\,\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(\sqrt{x}-y)^2}{2t}}\,dy = \frac{e^{-\frac{x}{2t}}}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(\lambda+\frac{1}{2t})y^2+\frac{\sqrt{x}}{t}y} \\ &= \frac{e^{-\frac{x}{2t}}}{\sqrt{2\pi t}}\sqrt{\frac{\pi}{\lambda+\frac{1}{2t}}}\,e^{\frac{x}{4t^2(\lambda+\frac{1}{2t})}} = \frac{1}{\sqrt{1+2\lambda t}}\,e^{-\frac{x}{2t}+\frac{x}{2t}\frac{1}{1+2\lambda t}} \\ &= \frac{1}{\sqrt{1+2\lambda t}}\,e^{-\frac{\lambda}{1+2\lambda t}x} \end{aligned} \tag{20.22}$$

which gives

$$B_\mu = \tfrac{1}{\sqrt{1+2\lambda t}}\,, \quad A_\mu = e^{-\frac{\lambda}{1+2\lambda t}} \tag{20.23}$$

and therefore

$$\int_0^\infty e^{-\lambda y}\, q_t^n(x,y)\, dy \;=\; \frac{1}{(1+2\lambda t)^{\frac{n}{2}}}\; e^{-\frac{\lambda}{1+2\lambda t}x} \tag{20.24}$$

However, this equation is solved by (20.12). ■

Next next theorem computes the numbers A_μ and B_μ in (20.11) of the last theorem. The measure is supposed to be a sum of a Dirac measure plus an absolute continuous part.

Theorem 20.2: Let $X_s = X_s^{0,x;n}$ be an n-dimensional squared Bessel process starting at $x \geq 0$ at time zero. Let $\rho = \rho(s)$ be some real density and $a \in \mathbb{R}$. Then

$$\mathsf{E}\Big[e^{\frac{a}{2}X_t - \frac{1}{2}\int_0^t X_s\rho(s)ds}\Big] \;=\; [f(t)]^{\frac{n}{2}}\, e^{\frac{1}{2}f'(0)x} \tag{20.25}$$

where the function $f(s)$ is the unique solution of the differential equation

$$f''(s) - \rho(s)\, f(s) = 0\,, \quad s \in (0,t) \tag{20.26}$$

with boundary conditions

$$f(0) \;=\; 1 \tag{20.27}$$

$$f'(t) - a\, f(t) \;=\; 0 \tag{20.28}$$

Proof: First we recall that, if X_t is some martingale, then also $Y_t = e^{X_t - \frac{1}{2}\langle X\rangle_t}$ is a martingale since, by Ito, $dY_t = Y_t(dX_t - \frac{1}{2}d\langle X\rangle_t) + \frac{1}{2}Y_t d\langle X\rangle_t = Y_t dX_t$. Now let X_t be the n-dimensional squared Bessel process starting at $x \geq 0$ at time zero. Then

$$M_t \;:=\; X_t - nt \tag{20.29}$$

is a martingale and, for some function $F = F(s)$, also

$$N_t \;:=\; \int_0^t F(s)\, dM_s \tag{20.30}$$

$$= \int_0^t F(s)\, dX_s - n\int_0^t F(s)\, ds$$

$$= F(t)X_t - F(0)X_0 - \int_0^t X_s dF(s) - n\int_0^t F(s)\, ds \tag{20.31}$$

is a martingale, and, by the above remark, also

$$\begin{aligned} Y_t &:= e^{N_t - \frac{1}{2}\langle N\rangle_t} \\ &= e^{\int_0^t F(s)\,dM_s - \frac{1}{2}\int_0^t F^2(s)d\langle M\rangle_s} \\ &\overset{(20.31)}{=} e^{F(t)X_t - F(0)x - \int_0^t X_s dF(s) - n\int_0^t F(s)\,ds - \frac{1}{2}\int_0^t F^2(s)d\langle X\rangle_s} \\ &\overset{(20.5)}{=} e^{F(t)X_t - F(0)x - \int_0^t X_s dF(s) - n\int_0^t F(s)\,ds - 2\int_0^t F^2(s)X_s\,ds} \\ &= e^{F(t)X_t - F(0)x - \int_0^t (F' + 2F^2)(s)X_s ds - n\int_0^t F(s)\,ds} \end{aligned} \tag{20.32}$$

is a martingale. Thus we have $\mathsf{E}[Y_t] = Y_0 = 1$ or,

$$\mathsf{E}\Big[e^{F(t)X_t - \int_0^t (F'+2F^2)(s)X_s ds}\Big] = e^{F(0)x + n\int_0^t F(s)\,ds} \tag{20.33}$$

Now let $F = \lambda f'/f = \lambda(\log f)'$ for some function f. Then

$$\begin{aligned} F' + 2F^2 &= \lambda\tfrac{f''f-(f')^2}{f^2} + 2\lambda^2\tfrac{(f')^2}{f^2} \\ &= \lambda\tfrac{f''}{f} + (2\lambda^2 - \lambda)\tfrac{(f')^2}{f^2} \\ &\overset{\lambda=\frac{1}{2}}{=} \frac{1}{2}\frac{f''}{f} \end{aligned} \tag{20.34}$$

and (20.33) becomes

$$\mathsf{E}\Big[e^{\frac{f'(t)}{2f(t)}X_t - \int_0^t \frac{f''}{2f}(s)X_s ds}\Big] = e^{\frac{f'(0)}{2f(0)}x + \frac{n}{2}\log(\frac{f(t)}{f(0)})} \tag{20.35}$$

Because of (20.26-20.28), the theorem follows. ■

Remark: By reviewing the above proof, one finds that for arbitrary *complex* f the following formula holds

$$\mathsf{E}\Big[e^{\frac{f'(t)}{2f(t)}X_t - \int_0^t \frac{f''}{2f}(s)X_s ds}\Big] = e^{\frac{f'(0)}{2f(0)}x + n\int_0^t \frac{f'(s)}{2f(s)}\,ds} \tag{20.36}$$

However, if f is complex we are not allowed to substitute the last integral in the exponent on the right hand side of (20.36) by $\log[f(t)/f(0)]$ or $\log f(t) - \log f(0)$, if the logarithm is meant to be the standard complex logarithm with imaginary part in, say, $(-\pi, \pi]$. For example, if $f(t) = e^{it}$, then the integral gives it whose imaginary part can be arbitrary large. However, by writing

$$f(s) = r(s)\,e^{i\varphi(s)} \tag{20.37}$$

with a differentiable φ (that is, φ is not restricted to an intervall of length 2π), we get

$$\begin{aligned} \int_0^t \frac{f'(s)}{f(s)}\,ds &= \int_0^t \frac{r'(s)\,e^{i\varphi(s)} + r(s)\,i\varphi'(s)\,e^{i\varphi(s)}}{r(s)\,e^{i\varphi(s)}}\,ds \\ &= \int_0^t \Big(\frac{r'(s)}{r(s)} + i\varphi'(s)\Big)ds \\ &= \log\Big[\frac{r(t)}{r(0)}\Big] + i\big(\varphi(t) - \varphi(0)\big) \end{aligned} \tag{20.38}$$

Thus we obtain the following complex case of Theorem 20.2 which we need later when we discuss the Cox-Ingersoll-Ross process and the Heston model.

Corollary 20.3: Let $X_s = X_s^{0,x;n}$ be an n-dimensional squared Bessel process starting at $x \geq 0$ at time zero. Let $a, c \geq 0$ and $b, d \in \mathbb{R}$, $\gamma = a + ib$ and $\xi = c + id$. Then

$$\mathsf{E}\Big[e^{\frac{\gamma}{2}X_t - \frac{\xi}{2}\int_0^t X_s\,ds}\Big] \;=\; e^{\frac{f'(0)}{f(0)}\frac{x}{2}+\frac{n}{2}\left\{\log[\frac{r_f(t)}{r_f(0)}]+i\left(\varphi_f(t)-\varphi_f(0)\right)\right\}} \tag{20.39}$$

where the function f is given by

$$\begin{aligned} f(s) &= \sqrt{\xi}\cosh[\sqrt{\xi}(t-s)] - \gamma\sinh[\sqrt{\xi}(t-s)] \\ &=: r_f(s)\,e^{i\varphi_f(s)} \end{aligned} \tag{20.40}$$

with a differentiable φ_f.

Proof: From the proof of Theorem 20.2 we find that for arbitrary complex f

$$\mathsf{E}\Big[e^{\frac{f'(t)}{2f(t)}X_t - \int_0^t \frac{f''}{2f}(s)X_s ds}\Big] \;=\; e^{\frac{f'(0)}{2f(0)}x + \frac{n}{2}\int_0^t \frac{f'(s)}{f(s)}\,ds} \tag{20.41}$$

Let

$$f(s) \;=\; \sqrt{\xi}\cosh[\sqrt{\xi}(t-s)] - \gamma\sinh[\sqrt{\xi}(t-s)] \tag{20.42}$$

Then $f'/(2f)(t) = \gamma/2$, $f''/f(s) = \xi$ for all s and the corollary is a consequence of (20.41) and (20.38). ■