Appendix 2

A2: American Options in the Black-Scholes Model

In lecture 4 we saw that the value of some european (exotic or not) option with payoff $C = C(\{S_t\}_{0\leq t\leq T})$ is given by

$$V_t(S_t) \;=\; e^{-r(T-t)}\mathsf{E}_{\tilde W}\big[\, C \,|\, S_t \,\big] \tag{2.1}$$

Furthermore we saw that, if $C = C(S_T)$, the function $V_t(S) = V(S,t)$ is a solution of the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.2}$$

with the final condition

$$V(S,T) = C(S) \tag{2.3}$$

The analog of (2.1) for american options is given by the continuous time version of Theorem 8.2 and is quite straight forward. If $U_t = U\big(\{S_u\}_{0\leq u\leq t}\big)$ is the option value at time t then

$$U_t \;=\; \sup_{\tau\geq t}\, \mathsf{E}_{\tilde W}[e^{-r(\tau-t)}\, C(\{S_u\}_{0\leq u\leq \tau}) \,|\, S_t\,] \tag{2.4}$$

$$\;=\; \mathsf{E}_{\tilde W}[e^{-r(\tau_{U=C}-t)}\, C(\{S_u\}_{0\leq u\leq \tau_{U=C}}) \,|\, S_t\,] \tag{2.5}$$

where $\tau_{U=C}$ is the stopping time defined by

$$\tau_{U=C} = \min\{ s\geq t \,|\, U_s = C_s\} \tag{2.6}$$

and $C_s = C(\{S_u\}_{0\leq u\leq s})$. Observe that in continuous time one cannot longer define U_t by backward induction as in (1.10,1.11).

We now want to consider the question whether there is an analog of (2.2) for american options whose payoff at time t is given by $C = C(S_t)$. To this end recall from lecture 8 the relation

$$v_k \;=\; u_k + \sum_{j=1}^{k}\Big(\max\big\{c_{j-1},\tilde u_{j-1}\big\} \;-\; \tilde u_{j-1}\Big) \tag{2.7}$$

Here u_k is the discounted Snell envelope for the american claim $C(S_{t_k})$ which is equal to the discounted option value at time t_k. v_k is the discounted portfolio value of a selffinancing strategy with initial value u_0,

$$v_k = u_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \tag{2.8}$$

where the δ's are given by (1.13). Finally $\tilde{u}_{j-1} = \mathsf{E}_{\tilde{B}}[u_j|s_{j-1}]$. From (2.7) we get

$$v_k - v_{k-1} = u_k - u_{k-1} + \Big(\max\{c_{k-1}, \tilde{u}_{k-1}\} \;-\; \tilde{u}_{k-1}\Big) \tag{2.9}$$

We write $v_k = v(S_{t_k}, t_k) = e^{-rt_k} V(S_{t_k}, t_k)$, $t_k = k\Delta t$ and consider the limit $\Delta t \to 0$. We have to distinguish two cases: For $u(S_{t_k}, t_k) > c(S_{t_k}, t_k)$ or $U(S_{t_k}, t_k) > C(S_{t_k})$ the round brackets in (2.9) vanish and we get

$$dv = du \tag{2.10}$$

This is equivalent to the Black-Scholes equation. Namely, because of (2.8),

$$dv = \delta\, ds = \tfrac{\partial u}{\partial s}\, ds = \tfrac{\partial U}{\partial S}\, d(e^{-rt}S) = \tfrac{\partial U}{\partial S}\big(-r\, e^{-rt} S dt + e^{-rt} dS\big) \tag{2.11}$$

and, using the Ito-Lemma for $U(S,t)$,

$$\begin{aligned} du &= d(e^{-rt}U) = -r\, e^{-rt} U dt + e^{-rt} dU \\ &= e^{-rt}\Big(-rU dt + \tfrac{\partial U}{\partial t} dt + \tfrac{\partial U}{\partial S} dS + \tfrac{1}{2}\tfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 dt\Big) \end{aligned} \tag{2.12}$$

Substituting this in (2.10),

$$\tfrac{\partial U}{\partial S}\big(-r\, S dt + dS\big) \;=\; -rU dt + \tfrac{\partial U}{\partial t} dt + \tfrac{\partial U}{\partial S} dS + \tfrac{1}{2}\tfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 dt \tag{2.13}$$

which is equivalent to

$$\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU \;=\; 0 \tag{2.14}$$

The second case is $U(S_{t_k}, t_k) = C(S_{t_k})$. In that case the round brackets in (2.9) are positive such that (2.10) has to be substituted by

$$dv \geq du \tag{2.15}$$

which gives, as above,

$$\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU \;\leq\; 0 \tag{2.16}$$

Thus we arrive at

$$\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU \leq 0\,, \qquad U(S,t) \geq C(S) \tag{2.17}$$

$$\Big(\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU\Big)\Big(U(S,t) - C(S)\Big) = 0 \tag{2.18}$$

$$U(S,T) = C(S) \tag{2.19}$$

In order to get a feeling for the system (2.17-2.19) we consider two examples which can be explicitely solved, the perpetual american put (example 9.1 below) and an american cash or nothing call (excercise 9.1). Perpetual means that the maturity date is $T = \infty$. The finite time $T < \infty$ american put is not explicitely solvable. The value of an american call coincides with the value of a european call.

Example 9.1 (perpetual american put): We consider an american put with payoff

$$C(S) = \max\{K - S, 0\} \equiv (K-S)_+ \tag{2.20}$$

which can be excercised at an arbitrary time $t \geq 0$, that is, its maturity is $T = \infty$. It is clear that its price cannot explicitely depend on t but only on $S = S_t$, the stock price at time t. Thus the system (2.17-2.19) reduces to

$$\tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 U}{\partial S^2} + rS\tfrac{\partial U}{\partial S} - rU \leq 0\,, \qquad U(S) \geq C(S) \tag{2.21}$$

$$\Big(\tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 U}{\partial S^2} + rS\tfrac{\partial U}{\partial S} - rU\Big)\Big(U(S) - C(S)\Big) = 0 \tag{2.22}$$

As in excercise 8.1 one can show that there is a stock price S_* such that $U(S) = C(S)$ for $S \leq S_*$ and $U(S) > C(S)$ for $S > S_*$. At $S = S_*$, U and U' has to be continuous. We first verify that $U(S) = C(S)$ satisfies the first inequality of (2.21). Indeed, since $S_* < K$, $C(S) = K - S$ and

$$\tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 C}{\partial S^2} + rS\tfrac{\partial C}{\partial S} - rC \;=\; 0 + rS(-1) - r(K-S) = -rK < 0 \tag{2.23}$$

For $S > S_*$, $U(S)$ has to be a solution of the Black-Scholes equation

$$\tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 U}{\partial S^2} + rS\tfrac{\partial U}{\partial S} - rU \;=\; 0 \tag{2.24}$$

The Ansatz $U(S) = S^\alpha$ gives

$$\begin{aligned} & \alpha(\alpha-1)\tfrac{\sigma^2}{2} + \alpha r - r \;=\; 0 \\ \Rightarrow \qquad & \alpha_1 = 1, \quad \alpha_2 = -\tfrac{2r}{\sigma^2} =: -\kappa \end{aligned} \tag{2.25}$$

Thus,

$$U(S) = c_1\, S + c_2\, S^{-\kappa} \tag{2.26}$$

Since we must have $U(S) \to 0$ as $S \to \infty$ it follows that $c_1 = 0$. Using the continuity of U and U' at S_* we find (recall $S_* < K$)

$$c_2\, S_*^{-\kappa} \;=\; K - S_* \tag{2.27}$$

$$-\kappa c_2\, S_*^{-\kappa-1} \;=\; -1 \tag{2.28}$$

Deviding (2.27) by (2.28) gives $-\frac{1}{\kappa} S_* = S_* - K$ or

$$S_* \;=\; \tfrac{\kappa}{\kappa+1}\, K \tag{2.29}$$

Furthermore $U(S) = c_2 S^{-\kappa} = c_2 S_*^{-\kappa}(S/S_*)^{-\kappa} \overset{2.27}{=} (K - S_*)(S/S_*)^{-\kappa}$. Hence we arrive at the following solution for the perpetual american put

$$U(S) = \begin{cases} K - S & \text{if } S \leq S_* \\ (K - S_*)(S/S_*)^{-\kappa} & \text{if } S > S_* \end{cases} \tag{2.30}$$

with $\kappa = 2r/\sigma^2$ and S_* given by (2.29).

Now we use formula (2.5) to compute $U(S)$ and show that the result coincides with (2.30). We have to compute

$$\begin{aligned} U(S) \;&=\; \mathsf{E}_{\tilde W}\big[e^{-r\tau}\, C(S_\tau^{(\mu)})\big] \\ &=\; \mathsf{E}_{W}\big[e^{-r\tau}\, C(S_\tau^{(r)})\big] \end{aligned} \tag{2.31}$$

where τ is the stopping time

$$\begin{aligned} \tau(\{S_t^{(r)}\}) \;&=\; \inf\big\{t \,|\, U(S_t^{(r)}) = C(S_t^{(r)})\big\} \\ &=\; \inf\big\{t \,|\, S_t^{(r)} \leq S_*\big\} \\ &=\; \inf\big\{t \,|\, S\, e^{\sigma x_t + (r - \frac{\sigma^2}{2})t} \leq S_*\big\} \\ &=\; \inf\big\{t \,|\, x_t + ct \leq \tfrac{1}{\sigma} \log \tfrac{S_*}{S}\big\} \end{aligned} \tag{2.32}$$

and we abbreviated $c = \frac{r}{\sigma} - \frac{\sigma}{2}$. For $S \leq S_*$ we have $\tau = 0$ and $U(S) = C(S)$ as in (2.30). Suppose $S > S_*$. Then $\tau > 0$ and $S_\tau^{(r)} = S_*$, hence $C(S_\tau^{(r)}) = C(S_*) = K - S_*$. Thus we have to compute

$$U(S) \;=\; (K - S_*)\mathsf{E}_W\big[e^{-r\tau}\big] \tag{2.33}$$

The expectation on the right hand side of (2.33) is computed in Lemma 9.2 below. We get

$$\begin{aligned} U(S) &= (K-S_*)e^{-|\frac{1}{\sigma}\log\frac{S_*}{S}|\sqrt{2r+c^2}+\frac{c}{\sigma}\log\frac{S_*}{S}} \\ &= (K-S_*)\left(\frac{S}{S_*}\right)^{-\frac{\sqrt{2r+c^2}}{\sigma}-\frac{c}{\sigma}} \\ &= (K-S_*)\left(\frac{S}{S_*}\right)^{-\kappa} \end{aligned} \tag{2.34}$$

since

$$\begin{aligned} \frac{\sqrt{2r+c^2}}{\sigma}+\frac{c}{\sigma} &= \frac{1}{\sigma}\left(\sqrt{2r+\frac{r^2}{\sigma^2}-r+\frac{\sigma^2}{4}}+\frac{r}{\sigma}-\frac{\sigma}{2}\right) \\ &= \frac{1}{\sigma}\left(\frac{r}{\sigma}+\frac{\sigma}{2}+\frac{r}{\sigma}-\frac{\sigma}{2}\right) = \frac{2r}{\sigma^2} = \kappa \end{aligned} \tag{2.35}$$

It remains to verify the value of S_*. To this end recall from (2.4,2.5) that the stopping time (2.32) is optimal. That is, a stopping time defined by some $\tilde{S}_* \neq S_*$ would give a lower expectation. In other words, we can determine S_* by differentiating $(K-S_*)(S_*/S)^\kappa$ with respect to S_* and putting the derivative equal to zero:

$$\begin{aligned} -(S_*/S)^\kappa+\kappa(K-S_*)(S_*/S)^{\kappa-1}\frac{1}{S} &= 0 \\ \Leftrightarrow \qquad -S_*+\kappa(K-S_*) &= 0 \\ \Leftrightarrow \qquad S_* = \frac{\kappa}{\kappa+1}K & \end{aligned} \tag{2.36}$$

and this coincides with (2.29). This completes our discussion of the american put with infinite maturity time.

Lemma 9.2: Let b, c be real numbers and $\lambda > 0$. Let E_W be the expectation with respect to the Wiener measure $dW^{(0,\infty)}(\{x_t\}_{t>0})$ and let x_t be a Brownian motion. Define the stopping time

$$\tau_b^c := \inf\{\, t \geq 0 \,|\, x_t + ct = b \,\} \tag{2.37}$$

with the convention $\inf\emptyset = \infty$. Then

$$\mathsf{E}_W\left[e^{-\lambda\tau_b^c}\right] = e^{-|b|\sqrt{2\lambda+c^2}+cb} \tag{2.38}$$

Proof: We first prove (2.38) for $c = 0$. To this end, consider the martingale $S_t = e^{\sigma x_t - \frac{\sigma^2}{2}t}$ with $\frac{\sigma^2}{2} = \lambda$. The stopping theorem states that for every bounded stopping time τ

$$\mathsf{E}_W[S_\tau] = \mathsf{E}_W[S_0] = S_0 = 1 \tag{2.39}$$

Since τ_b^0 is not necessarily finite, one has to consider $\tau_{b,t}^0 := \min\{\tau_b^0, t\}$ and apply Lebesgue's theorem (assume without loss of generality $b > 0$ if $\sigma > 0$) to obtain

$$\begin{aligned} \mathsf{E}_W[S_{\tau_b^0}] &= \lim_{t\to\infty} \mathsf{E}_W[S_{\tau_{b,t}^0}] \\ &= \lim_{t\to\infty} \mathsf{E}_W[S_0] = S_0 = 1 \end{aligned} \tag{2.40}$$

Since $x_{\tau_b^0} = b$ we have $S_{\tau_b^0} = e^{\sigma b} e^{-\frac{\sigma^2}{2}\tau_b^0}$ and therefore

$$\mathsf{E}_W[e^{-\frac{\sigma^2}{2}\tau_b^0}] = e^{-\sigma b} = e^{-\sqrt{2\lambda}\,|b|} \tag{2.41}$$

which proves (2.38) for $c = 0$. The case $c \neq 0$ is obtained with Girsanov's theorem. We make the substitution of variables $y_s = x_s + cs$ and write $\tilde\tau_b^c(\{y_s\}) = \tau_b^c(\{x_s\}) = \tau_b^c(\{y_s - cs\}) = \tau_b^0(\{y_s\})$ to obtain

$$\begin{aligned} \mathsf{E}_W\left[e^{-\lambda\tau_b^c}\right] &= \int e^{-\lambda\tau_b^c(\{x_s\})} dW^{(0,\infty)}(\{x_s\}_{s>0}) \\ &= \int e^{-\lambda\tilde\tau_b^c(\{y_s\})} e^{cx_{\tau_b^c} + \frac{c^2}{2}\tau_b^c(\{x_s\})} dW^{(0,\infty)}(\{y_s\}_{s>0}) \\ &= \int e^{-\lambda\tau_b^0(\{y_s\})} e^{cy_{\tau_b^0} - \frac{c^2}{2}\tau_b^0(\{y_s\})} dW^{(0,\infty)}(\{y_s\}_{s>0}) \\ &= e^{cb} \int e^{-(\lambda + \frac{c^2}{2})\tau_b^0(\{y_s\})} dW^{(0,\infty)}(\{y_s\}_{s>0}) \\ &\overset{(2.41)}{=} e^{cb}\, e^{-|b|\sqrt{2\lambda + c^2}} \end{aligned} \tag{2.42}$$

This proves the lemma ■

Example 9.2 (american cash-or-nothing call): The cash-or-nothing call has a payoff

$$C(S_t) = \chi(S_t \geq K) = \begin{cases} 1 & \text{falls } S_t \geq K \\ 0 & \text{falls } S_t < K \end{cases} \tag{2.43}$$

at time t. For the price process we assume a geometric Brownian motion. We consider first the european case, then we compute the price of the american claim using the system (2.17-2.19) and finally we use (2.5) to calculate the price.

(i) The european case: According to (2.1), we get, using Lemma 6.7 in the second line and abbreviating $c = \frac{r}{\sigma} - \frac{\sigma}{2}$, $\theta = T - t$ (τ is reserved for a stopping time)

$$\begin{aligned} V_t(S_t) &= e^{-r(T-t)} \mathsf{E}_{\tilde W}\left[\, C(S_T \,|\, S_t\,\right] \\ &= e^{-r(T-t)} \int_{\mathbb{R}} \chi\big(S_t\, e^{\sigma y_{T-t} + (r - \frac{\sigma^2}{2})(T-t)} \geq K\big)\, p_{T-t}(0, y_{T-t})\, dy_{T-t} \\ &= e^{-r\theta} \int_{\mathbb{R}} \chi\big(y_\theta + c\theta \geq \tfrac{1}{\sigma}\log\tfrac{K}{S_t}\big)\, \tfrac{1}{\sqrt{2\pi\theta}}\, e^{-\frac{y_\theta^2}{2\theta}} dy_\theta \\ &= e^{-r\theta}\left\{1 - N\left(\frac{\frac{1}{\sigma}\log\frac{K}{S_t} - c\tau}{\sqrt{\theta}}\right)\right\} \end{aligned} \tag{2.44}$$

(ii) The american case, using the Black-Scholes equation: We have to solve the system

$$\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU \leq 0\,, \qquad U(S,t) \geq \chi(S \geq K) \tag{2.45}$$

$$\Big(\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU\Big)\Big(U(S,t) - \chi(S \geq K)\Big) = 0 \tag{2.46}$$

$$U(S,T) = \chi(S \geq K) \tag{2.47}$$

Since $U(S,t) > 0$ for all $t \in [0,T)$, we have

$$\begin{aligned} U(S,t) \;=\; \chi(S \geq K) \qquad &\Leftrightarrow \qquad U(S,t) \;=\; 1 \;=\; \chi(S \geq K) \\ &\Leftrightarrow \qquad S \geq K \end{aligned} \tag{2.48}$$

Furthermore we have the boundary condition

$$U(S,t) \to 0 \;\text{ for }\; S \to 0 \tag{2.49}$$

In excercise 9.1 it has been shown that the transformation ($\kappa = 2r/\sigma^2$)

$$S = Ke^x, \quad t = T - 2\tau/\sigma^2, \quad (x,\tau) \in \mathbb{R} \times [0, \sigma^2 T/2] \tag{2.50}$$

and

$$u(x,\tau) := \frac{1}{K}\, e^{\frac{1}{2}(\kappa-1)x + \frac{1}{4}(\kappa+1)^2\tau} U(S,t) \tag{2.51}$$

leads to the system

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0\,, \qquad u(x,\tau) \geq g(x,\tau) \tag{2.52}$$

$$\Big(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\Big)\Big(u(x,\tau) - g(x,\tau)\Big) = 0 \tag{2.53}$$

$$u(x,0) = g(x,0) \tag{2.54}$$

where

$$g(x,\tau) = \frac{1}{K}\, e^{\frac{1}{2}(\kappa-1)x + \frac{1}{4}(\kappa+1)^2\tau} \chi(x \geq 0) \tag{2.55}$$

Because of (2.48) we have

$$u(x,\tau) = g(x,\tau) \quad \text{for all } x \geq 0,\ \tau \geq 0 \tag{2.56}$$

and $u(x,\tau) > g(x,\tau)$ for $x < 0$. Hence, for negative x, the function u has to be a solution of the diffusion equation

$$\frac{\partial u}{\partial \tau} \;=\; \frac{\partial^2 u}{\partial x^2}\,, \quad \text{for } x < 0 \tag{2.57}$$

$$u(x,0) \;=\; 0\,, \quad \text{for } x < 0 \tag{2.58}$$

with boundary conditions

$$u(0,\tau) = \tfrac{1}{K}\, e^{\frac{1}{4}(\kappa+1)^2\tau} \tag{2.59}$$

$$\lim_{x\to-\infty} u(x,\tau) = 0 \tag{2.60}$$

Such a system can be solved with a Laplace transformation with respect to τ. Let

$$v(x,\omega) := \int_0^\infty e^{-\omega\tau} u(x,\tau)\, d\tau \tag{2.61}$$

Then (2.57) becomes

$$\omega\, v(x,\omega) - u(x,0) = \omega\, v(x,\omega) = \tfrac{d^2 v}{dx^2}(x,\omega) \tag{2.62}$$

with general solution

$$v(x,\omega) = a_\omega\, e^{\sqrt{\omega}x} + b_\omega\, e^{-\sqrt{\omega}x} \tag{2.63}$$

Because of (2.60) the coefficient $a_\omega \equiv 0$ and the boundary condition (2.59) gives

$$\begin{aligned} b_\omega &= v(0,\omega) \\ &= \textstyle\int_0^\infty e^{-\omega\tau} u(0,\tau)\, d\tau \\ &=: \mathcal{L}[u(0,\,\cdot\,)](\omega) \end{aligned} \tag{2.64}$$

Let $f(\tau) = f_x(\tau)$ be the function whose Laplace transform is $e^{-x\sqrt{\omega}}$,

$$\mathcal{L}[f_x](\omega) = e^{-x\sqrt{\omega}} \tag{2.65}$$

Then, by the properties of the Laplace transform,

$$\begin{aligned} v(x,\omega) &= b_\omega\, e^{-\sqrt{\omega}x} \\ &= \mathcal{L}[u(0,\,\cdot\,)](\omega)\, \mathcal{L}[f_x](\omega) \\ &= \mathcal{L}[u(0,\,\cdot\,) * f_x](\omega) \end{aligned} \tag{2.66}$$

such that

$$\begin{aligned} u(x,\tau) &= u(0,\,\cdot\,) * f_x(\tau) \\ &= \textstyle\int_0^\tau u(0,\tau - s)\, f_x(s)\, ds \end{aligned} \tag{2.67}$$

It remains to determine the function f_x. To this end we use the following integral [7], 3.471.9:

$$\textstyle\int_0^\infty \tau^{\nu-1} e^{-\gamma\tau - \frac{\beta}{\tau}} d\tau = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}) \tag{2.68}$$

for Re$\beta > 0$, Re$\gamma > 0$. Here $K_\nu = K_{-\nu}$ is a modified Bessel function [7], 8.407. We only need the values for $\nu = n + \frac{1}{2}$. In that case [7], 8.468:

$$K_{n+\frac{1}{2}}(z) = \sqrt{\tfrac{\pi}{2z}}\, e^{-z} \sum_{k=0}^{n} \tfrac{(n+k)!}{(n-k)!\,k!}\,(2z)^{-k} \tag{2.69}$$

In particular,

$$\begin{aligned}\int_0^\infty \tau^{-\frac{1}{2}-1} e^{-\omega\tau - \frac{x^2/4}{\tau}} d\tau &= 2\big(\tfrac{x^2/4}{\omega}\big)^{-\frac{1}{4}} K_{-\frac{1}{2}}(2\sqrt{\omega x^2/4}) \\ &= 2\tfrac{\sqrt{2}\,\omega^{\frac{1}{4}}}{\sqrt{x}}\, K_{\frac{1}{2}}(\sqrt{\omega} x) \\ &= 2\tfrac{\sqrt{2}\,\omega^{\frac{1}{4}}}{\sqrt{x}} \sqrt{\tfrac{\pi}{2\sqrt{\omega}x}}\, e^{-\sqrt{\omega}x} \\ &= \tfrac{\sqrt{4\pi}}{x}\, e^{-\sqrt{\omega}x} \end{aligned} \tag{2.70}$$

which gives

$$f_x(\tau) = \tfrac{x}{\sqrt{4\pi\tau^3}}\, e^{-\frac{x^2}{4\tau}} \tag{2.71}$$

Thus, recalling (2.67), we end up with

$$u(x,\tau) = \int_0^\tau \tfrac{1}{K}\, e^{\frac{1}{4}(\kappa+1)^2(\tau-s)}\, \tfrac{x}{\sqrt{4\pi s^3}}\, e^{-\frac{x^2}{4s}}\, ds \tag{2.72}$$

or, recalling $x = \log(S/K)$,

$$\begin{aligned} U(S,t) &= Ke^{-\frac{1}{2}(\kappa-1)x - \frac{1}{4}(\kappa+1)^2\tau} u(x,\tau) \\ &= e^{-\frac{1}{2}(\kappa-1)x} \int_0^\tau e^{-\frac{1}{4}(\kappa+1)^2 s}\, \tfrac{x}{\sqrt{4\pi s^3}}\, e^{-\frac{x^2}{4s}}\, ds \end{aligned} \tag{2.73}$$

(iii) The american case, using stopping times: According to (2.5), we have

$$U(S_t,t) = \mathsf{E}_{\tilde{W}}\big[e^{-r(\tau_{U=C}-t)}\, C(S_{\tau_{U=C}})\,|\,S_t\big] \tag{2.74}$$

where $\tau_{U=C}$ is the stopping time given by $\tau_{U=C} = \inf\big\{u \geq t\,|\,U(S_u) = C(S_u)\big\}$. Apparently, $U \leq 1$ and $U = 1 \Leftrightarrow S_u \geq K$. By applying again Lemma 6.7, we have to compute an integral with respect to the Wiener measure whose integrand is a function of $\{S_t\, e^{\sigma y_s + (r - \frac{\sigma^2}{2})s}\}_{0<s\leq T-t}$. Hence, the condition $S_\tau \geq K$ becomes ($x = \log(S/K)$)

$$y_\tau + c\tau \;\geq\; \tfrac{1}{\sigma}\log\tfrac{K}{S_t} = -\tfrac{x}{\sigma} \tag{2.75}$$

and the stopping time $\tau_{U=C}$, because $U > 0$ for $t < T$, is given by

$$\begin{aligned} \tau_{U=C} &= \inf_{u\in[t,T]} \left\{ U(S_u) = C(S_u) \right\} \\ &= \inf_{u\in[t,T]} \left\{ U(S_u) = 1 = C(S_u) \right\} \\ &= \inf_{u\in[t,T]} \left\{ S_u = K \right\} \\ &= \inf_{u\in[t,T]} \left\{ y_{u-t} + c(u-t) = -\tfrac{x}{\sigma} \right\} \end{aligned} \tag{2.76}$$

or, abbreviating $b = -\frac{x}{\sigma}$ and $\theta = T - t$,

$$\begin{aligned} \tau_b^c &:= \tau_{U=C} - t \\ &= \inf_{s\in[0,T-t]} \left\{ y_s + cs = b \right\} \end{aligned} \tag{2.77}$$

Thus,

$$\begin{aligned} U(S_t,t) &= \mathsf{E}_{\tilde{W}}\left[e^{-r(\tau_{U=C}-t)}\,\chi(S_{\tau_{U=C}} \geq K)\,|\,S_t\right] \\ &= \int e^{-r\tau_b^c}\,\chi\big(\tau_b^c \leq \theta\big)\,dW(\{y_s\}_{0<s\leq\theta}) \\ &\overset{\text{Girsanov}}{=} \int e^{-r\tau_b^0}\,\chi\big(\tau_b^0 \leq \theta\big)\,e^{cx_{\tau_b^0}-\frac{c^2}{2}\tau_b^0}\,dW(\{x_s\}_{0<s\leq\theta}) \\ &= e^{cb}\int e^{-(r+\frac{c^2}{2})\tau_b^0}\,\chi\big(\tau_b^0 \leq \theta\big)\,dW(\{x_s\}_{0<s\leq\theta}) \\ &= e^{-\frac{c}{\sigma}x}\int_0^\theta e^{-(r+\frac{1}{2}(\frac{r}{\sigma}-\frac{\sigma}{2})^2)u}\,\mathsf{P}\big(\tau_b^0 \in [u,u+du)\big) \\ &= e^{-\frac{1}{2}(\kappa-1)x}\int_0^\theta e^{-ru+\frac{1}{2}(\frac{r^2}{\sigma^2}-r+\frac{\sigma^2}{4})u}\,\tfrac{d}{du}\mathsf{P}\big(\tau_b^0 \leq u\big)\,du \\ &\overset{(10.21)}{=} e^{-\frac{1}{2}(\kappa-1)x}\int_0^\theta e^{-\frac{1}{2}(\frac{r^2}{\sigma^2}+r+\frac{\sigma^2}{4})u}\,\tfrac{d}{du}2\int_{b/\sqrt{u}}^\infty \tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}}\,dy \\ &= e^{-\frac{1}{2}(\kappa-1)x}\int_0^{T-t} e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2u}\,\tfrac{-b}{\sqrt{u}^3}\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{b^2}{2u}}du \\ &\overset{u=\frac{2s}{\sigma^2}}{=} e^{-\frac{1}{2}(\kappa-1)x}\int_0^{\sigma^2(T-t)/2} e^{-(\frac{r}{\sigma^2}+\frac{1}{2})^2s}\,\tfrac{(x/\sigma)\sigma^3}{\sqrt{2s}^3}\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{4s}}\,\tfrac{2}{\sigma^2}\,ds \\ &= e^{-\frac{1}{2}(\kappa-1)x}\int_0^{\sigma^2(T-t)/2} e^{-\frac{1}{4}(\kappa+1)^2s}\,\tfrac{x}{\sqrt{4\pi s^3}}\,e^{-\frac{x^2}{4s}}\,ds \end{aligned} \tag{2.78}$$

and this coincides with the solution (2.73) of the PDE-system (2.45-2.47) since $2\tau/\sigma^2 = T - t$.