Hochschule RheinMain
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Lösungen 8. Übungsblatt
Mathematische Methoden in der Quantenmechanik

Aufgabe 1: a) Wir haben für eine symmetrische n-Teilchen Wellenfunktion $F_n \in L^2_s(\Gamma^n)$:

$$\begin{aligned}
(a_\alpha a_\beta^+ F_n)(x_1,\cdots,x_n) &\overset{\text{Thm 8.1}}{=} \sqrt{n+1}\,\sum_{x_0} \bar{e}_\alpha(x_0)\,(a_\beta^+ F_n)(x_0,x_1,\cdots,x_n) \\
&\overset{\text{Def 8.1}}{=} (n+1)\,\sum_{x_0} \bar{e}_\alpha(x_0)\,(e_\beta \otimes_s F_n)(x_0,x_1,\cdots,x_n) \\
&\overset{\text{Lemma 8.1}}{=} (n+1)\,\sum_{x_0} \bar{e}_\alpha(x_0)\,\frac{1}{n+1}\sum_{i=1}^{n+1} e_\beta(x_{i-1})\,F_n(x_0,\cdots,\widehat{x_{i-1}},\cdots,x_n) \\
&= \sum_{x_0} \bar{e}_\alpha(x_0)\,\sum_{i=1}^{n+1} e_\beta(x_{i-1})\,F_n(x_0,\cdots,\widehat{x_{i-1}},\cdots,x_n) \\
&= (e_\alpha,e_\beta)\,f_n(x_1,\cdots,x_n) \;+\; \sum_{x_0} \bar{e}_\alpha(x_0)\,\sum_{i=2}^{n+1} e_\beta(x_{i-1})\,F_n(x_0,\cdots,\widehat{x_{i-1}},\cdots,x_n)
\end{aligned}$$

oder

$$\begin{aligned}
&(a_\alpha a_\beta^+ F_n)(x_1,\cdots,x_n) \\
&\quad= (e_\alpha,e_\beta)\,F_n(x_1,\cdots,x_n) \;+\; \sum_{x_0} \bar{e}_\alpha(x_0)\,\sum_{i=1}^{n} e_\beta(x_i)\,F_n(x_0,\cdots,\widehat{x_i},\cdots,x_n) \\
&\quad= (e_\alpha,e_\beta)\,F_n(x_1,\cdots,x_n) \;+\; \sum_{i=1}^{n} e_\beta(x_i)\,\sum_{x_0} \bar{e}_\alpha(x_0)\,F_n(x_0,\cdots,\widehat{x_i},\cdots,x_n)
\end{aligned}$$

Andererseits,

$$\begin{aligned}
(a_\beta^+ a_\alpha F_n)(x_1,\cdots,x_n) &\overset{\text{Def 8.1}}{=} \sqrt{n}\,(e_\beta \otimes_s [a_\alpha F_n])(x_1,\cdots,x_n) \\
&\overset{\text{Lemma 8.1}}{=} \sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n} e_\beta(x_i)\,[a_\alpha F_n](x_1,\cdots,\widehat{x_i},\cdots,x_n) \\
&\overset{\text{Thm 8.1}}{=} \sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n} e_\beta(x_i)\,\sqrt{n}\,\sum_{x_0} \bar{e}_\alpha(x_0)\,F_n(x_0,x_1,\cdots,\widehat{x_i},\cdots,x_n) \\
&= \sum_{i=1}^{n} e_\beta(x_i)\,\sum_{x_0} \bar{e}_\alpha(x_0)\,F_n(x_0,x_1,\cdots,\widehat{x_i},\cdots,x_n)
\end{aligned}$$

Also

$$a_\alpha a_\beta^+ F_n \;-\; a_\beta^+ a_\alpha F_n \;=\; (e_\alpha,e_\beta)\,F_n \;=\; \delta_{\alpha,\beta}\,F_n \;\;.$$

b) Im fermionischen Fall bekommen wir für eine beliebige antisymmetrische n-Teilchen Wellenfunktion $G_n \in L^2_a(\Gamma^n)$:

$$
\begin{aligned}
(c_\alpha c_\beta^+ G_n)(x_1, \cdots, x_n) \;&=\; \sqrt{n+1}\, \sum_{x_0} \bar{e}_\alpha(x_0)\, (c_\beta^+ G_n)(x_0, x_1, \cdots, x_n) \\
&=\; (n+1) \sum_{x_0} \bar{e}_\alpha(x_0)\, (e_\beta \otimes_a G_n)(x_0, x_1, \cdots, x_n) \\
&=\; (n+1) \sum_{x_0} \bar{e}_\alpha(x_0)\, \frac{1}{n+1} \sum_{i=1}^{n+1} (-1)^{i-1}\, e_\beta(x_{i-1})\, G_n(x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n) \\
&=\; \sum_{x_0} \bar{e}_\alpha(x_0) \sum_{i=1}^{n+1} (-1)^{i-1}\, e_\beta(x_{i-1})\, G_n(x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n) \\
&=\; (e_\alpha, e_\beta)\, G_n(x_1, \cdots, x_n) \;+\; \sum_{x_0} \bar{e}_\alpha(x_0) \sum_{i=2}^{n+1} (-1)^{i-1}\, e_\beta(x_{i-1})\, G_n(x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n)
\end{aligned}
$$

oder

$$
\begin{aligned}
&(c_\alpha c_\beta^+ G_n)(x_1, \cdots, x_n) \\
&\quad=\; (e_\alpha, e_\beta)\, G_n(x_1, \cdots, x_n) \;+\; \sum_{x_0} \bar{e}_\alpha(x_0) \sum_{i=1}^{n} (-1)^{i}\, e_\beta(x_i)\, G_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n) \\
&\quad=\; (e_\alpha, e_\beta)\, G_n(x_1, \cdots, x_n) \;+\; \sum_{i=1}^{n} (-1)^{i}\, e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, G_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n)
\end{aligned}
$$

Andererseits,

$$
\begin{aligned}
(c_\beta^+ c_\alpha G_n)(x_1, \cdots, x_n) \;&=\; \sqrt{n}\, (e_\beta \otimes_a [c_\alpha G_n])(x_1, \cdots, x_n) \\
&=\; \sqrt{n}\, \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1}\, e_\beta(x_i)\, [c_\alpha G_n](x_1, \cdots, \widehat{x}_i, \cdots, x_n) \\
&=\; \sqrt{n}\, \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1}\, e_\beta(x_i)\, \sqrt{n} \sum_{x_0} \bar{e}_\alpha(x_0)\, G_n(x_0, x_1, \cdots, \widehat{x}_i, \cdots, x_n) \\
&=\; -\sum_{i=1}^{n} (-1)^{i}\, e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, G_n(x_0, x_1, \cdots, \widehat{x}_i, \cdots, x_n)
\end{aligned}
$$

Also

$$
c_\alpha c_\beta^+ G_n \;+\; c_\beta^+ c_\alpha G_n \;=\; (e_\alpha, e_\beta)\, G_n \;=\; \delta_{\alpha,\beta}\, G_n \;.
$$

Aufgabe 2: Aus Aufgabe 1 haben wir die Formeln

$$
\begin{aligned}
(a_\beta^+ a_\alpha F_n)(x_1, \cdots, x_n) \;&=\; \sum_{i=1}^{n} e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, F_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n) \\
(c_\beta^+ c_\alpha G_n)(x_1, \cdots, x_n) \;&=\; -\sum_{i=1}^{n} (-1)^{i}\, e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, G_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n)
\end{aligned}
$$

In der x_0-Summe tun wir in beiden Fällen das x_0 durch ein y_i ersetzen. Das F_n hat dann die Argument-Liste

$$\begin{aligned} F_n &= F_n(x_0, x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_n) \\ &= F_n(y_i, x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_n) \\ &= F_n(x_1, \cdots, x_{i-1}, y_i, x_{i-1}, \cdots, x_n) \end{aligned}$$

wobei wir beim letzten Gleichheitszeichen die Symmetrie des F_n verwendet haben. Für das G_n, das ist antisymmetrisch, erhalten wir

$$\begin{aligned} G_n &= G_n(x_0, x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_n) \\ &= G_n(y_i, x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_n) \\ &= (-1)^{i-1}\, G_n(x_1, \cdots, x_{i-1}, y_i, x_{i-1}, \cdots, x_n) \end{aligned}$$

Also,

$$\begin{aligned} (a_\beta^+ a_\alpha F_n)(x_1, \cdots, x_n) &= \sum_{i=1}^n e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, F_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n) \\ &= \sum_{i=1}^n e_\beta(x_i) \sum_{y_i} \bar{e}_\alpha(y_i)\, F_n(y_i, x_1, \cdots, \widehat{x}_i, \cdots, x_n) \\ &= \sum_{i=1}^n \sum_{y_i \in \Gamma} e_\beta(x_i)\, \bar{e}_\alpha(y_i)\, F_n(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \end{aligned}$$

und

$$\begin{aligned} (c_\beta^+ c_\alpha G_n)(x_1, \cdots, x_n) &= -\sum_{i=1}^n (-1)^i\, e_\beta(x_i) \sum_{x_0} \bar{e}_\alpha(x_0)\, G_n(x_0, \cdots, \widehat{x}_i, \cdots, x_n) \\ &= \sum_{i=1}^n (-1)^{i-1}\, e_\beta(x_i) \sum_{y_i} \bar{e}_\alpha(y_i)\, G_n(y_i, x_1, \cdots, \widehat{x}_i, \cdots, x_n) \\ &= \sum_{i=1}^n (-1)^{i-1}\, e_\beta(x_i) \sum_{y_i} \bar{e}_\alpha(y_i)\, (-1)^{i-1}\, G_n(x_1, \cdots, x_{i-1}, y_i, x_{i-1}, \cdots, x_n) \\ &= \sum_{i=1}^n \sum_{y_i} e_\beta(x_i)\bar{e}_\alpha(y_i)\, G_n(x_1, \cdots, x_{i-1}, y_i, x_{i-1}, \cdots, x_n) \end{aligned}$$

Da nach Voraussetzung die $\{e_\alpha\}$ eine ONB bilden, gilt

$$\sum_\alpha \bar{e}_\alpha(x) e_\alpha(y) = \delta_{x,y}$$

Also,

$$\begin{aligned} \sum_\alpha (a_\alpha^+ a_\alpha F_n)(x_1, \cdots, x_n) &= \sum_{i=1}^n \sum_{y_i \in \Gamma} \sum_\alpha e_\alpha(x_i)\bar{e}_\alpha(y_i)\; F_n(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \\ &= \sum_{i=1}^n \sum_{y_i \in \Gamma} \delta_{x_i, y_i}\; F_n(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \\ &= \sum_{i=1}^n F_n(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \\ &= n\, F_n(x_1, \cdots, x_n) \end{aligned}$$

mit derselben Rechnung für den fermionischen Fall.