

Lösungen zum 9. Übungsblatt
Einführung in die Finanzmathematik

Aufgabe 1: a) Wir haben

$$N - k = N - \frac{N + \sqrt{N}x}{2} = \frac{N - \sqrt{N}x}{2}$$

und

$$\begin{aligned} \frac{1}{2^N} \binom{N}{k} &= \frac{1}{2^N} \frac{N!}{k!(N-k)!} \\ &\stackrel{\text{Stirling}}{\approx} \frac{1}{2^N} \frac{\sqrt{2\pi N} \left(\frac{N}{e}\right)^N}{\sqrt{2\pi k} \left(\frac{k}{e}\right)^k \sqrt{2\pi(N-k)} \left(\frac{N-k}{e}\right)^{N-k}} \\ &= \frac{1}{2^N} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{N}{k(N-k)}} \frac{N^N}{k^k(N-k)^{N-k}} \end{aligned}$$

b)

$$\begin{aligned} \frac{N}{k} &= \frac{N}{\frac{N+\sqrt{N}x}{2}} = \frac{2}{1 + \frac{x}{\sqrt{N}}}, \\ \frac{N}{N-k} &= \frac{N}{\frac{N-\sqrt{N}x}{2}} = \frac{2}{1 - \frac{x}{\sqrt{N}}} \end{aligned}$$

c)

$$\begin{aligned} \frac{1}{2^N} \frac{N^N}{k^k(N-k)^{N-k}} &= \frac{1}{2^N} \left(\frac{N}{k}\right)^k \left(\frac{N}{N-k}\right)^{N-k} \\ &= \frac{1}{2^N} \left(\frac{2}{1 + \frac{x}{\sqrt{N}}}\right)^k \left(\frac{2}{1 - \frac{x}{\sqrt{N}}}\right)^{N-k} \\ &= \left(\frac{1}{1 + \frac{x}{\sqrt{N}}}\right)^k \left(\frac{1}{1 - \frac{x}{\sqrt{N}}}\right)^{N-k} \\ &= \left(\frac{1}{1 + \frac{x}{\sqrt{N}}}\right)^{\frac{N+\sqrt{N}x}{2}} \left(\frac{1}{1 - \frac{x}{\sqrt{N}}}\right)^{\frac{N-\sqrt{N}x}{2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\left(1 - \frac{x^2}{N}\right)^{N/2}} \times \frac{\left(1 - \frac{x}{\sqrt{N}}\right)^{\frac{\sqrt{N}x}{2}}}{\left(1 + \frac{x}{\sqrt{N}}\right)^{\frac{\sqrt{N}x}{2}}} \\
&\xrightarrow{N \rightarrow \infty} \frac{1}{e^{-\frac{x^2}{2}}} \times \frac{(e^{-x})^{x/2}}{(e^x)^{x/2}} = e^{-\frac{x^2}{2}}
\end{aligned}$$

d) Schliesslich, mit

$$\Delta x := x_{N,k} - x_{N,k-1} = \frac{2k - N}{\sqrt{N}} - \frac{2(k-1) - N}{\sqrt{N}} = \frac{2}{\sqrt{N}}$$

bekommen wir

$$\begin{aligned}
\frac{\sqrt{\frac{N}{k(N-k)}}}{\Delta x} &= \frac{\sqrt{\frac{N}{\left(\frac{N+\sqrt{N}x}{2}\right)\left(\frac{N-\sqrt{N}x}{2}\right)}}}{\frac{2}{\sqrt{N}}} \\
&= \frac{\sqrt{\frac{4N}{N^2 - Nx^2}}}{\frac{2}{\sqrt{N}}} \\
&= \sqrt{\frac{N^2}{N^2 - Nx^2}} \\
&= \sqrt{\frac{1}{1 - \frac{x^2}{N}}} \xrightarrow{N \rightarrow \infty} 1
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{2^N} \binom{N}{k} &\approx \frac{1}{\sqrt{2\pi}} \times \frac{1}{2^N} \frac{N^N}{k^k (N-k)^{N-k}} \times \sqrt{\frac{N}{k(N-k)}} \\
&\approx \frac{1}{\sqrt{2\pi}} \times e^{-\frac{x^2}{2}} \times \Delta x
\end{aligned}$$

Aufgabe 2: a) Es ist

$$\begin{aligned}
\int_0^\infty x^N e^{-x} dx &= x^N e^{-x} \Big|_0^\infty - \int_0^\infty N x^{N-1} (-e^{-x}) dx \\
&= N \int_0^\infty x^{N-1} e^{-x} dx \\
&= N(N-1) \int_0^\infty x^{N-2} e^{-x} dx \\
&\vdots \\
&= N(N-1) \cdots 2 \cdot 1 \int_0^\infty x^0 e^{-x} dx = N!
\end{aligned}$$

b) Mit der Substitution $x = y + N$ bekommen wir

$$\begin{aligned} N! &= \int_0^{\infty} x^N e^{-x} dx \\ &= \int_{-N}^{\infty} (y + N)^N e^{-y-N} dy \\ &= \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} \left(\frac{y+N}{N}\right)^N e^{-y} dy \\ &= \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} e^{N \log[1 + \frac{y}{N}] - y} dy \end{aligned}$$

c) Und mit der Taylor-Entwicklung $\log(1 + h) \approx h - \frac{h^2}{2}$ erhalten wir

$$\begin{aligned} N! &= \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} e^{N \log[1 + \frac{y}{N}] - y} dy \\ &\approx \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} e^{N \left[\frac{y}{N} - \frac{1}{2} \left(\frac{y}{N}\right)^2\right] - y} dy \\ &= \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} e^{-\frac{y^2}{2N}} dy \end{aligned}$$

d)

$$\begin{aligned} N! &\approx \left(\frac{N}{e}\right)^N \int_{-N}^{\infty} e^{-\frac{y^2}{2N}} dy \\ &= \left(\frac{N}{e}\right)^N \int_{-\sqrt{N}}^{\infty} e^{-\frac{z^2}{2}} dz \sqrt{N} \\ &= \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \times \int_{-\sqrt{N}}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\ &\approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \times \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\ &= \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \times 1. \end{aligned}$$