Hochschule RheinMain
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**Lösungen zum 7. Übungsblatt**
**Einführung in die Finanzmathematik**

**1. Aufgabe:** Nach Theorem 5.2 gilt

$$V_0 \;=\; (1+r)^{-N} \times \sum_{\ell=0}^{N} H(S_{N,\ell}) \times \binom{N}{\ell} \, p_{\rm rn}^{\ell} \, (1-p_{\rm rn})^{N-\ell}$$

mit den Underlyingpreisen

$$S_{N,\ell} \;=\; S_0 \, (1 + {\rm ret_{up}})^{\ell} \, (1 + {\rm ret_{down}})^{N-\ell}$$

Wir haben dann

$$H(S_{N,\ell}) \;=\; \frac{S_0}{S_{N,\ell}} \;=\; \frac{1}{(1 + {\rm ret_{up}})^{\ell} \, (1 + {\rm ret_{down}})^{N-\ell}}$$

und bekommen damit

$$\begin{aligned} V_0 \;&=\; (1+r)^{-N} \times \sum_{\ell=0}^{N} \frac{1}{(1 + {\rm ret_{up}})^{\ell} \, (1 + {\rm ret_{down}})^{N-\ell}} \times \binom{N}{\ell} \, p_{\rm rn}^{\ell} \, (1-p_{\rm rn})^{N-\ell} \\ &=\; (1+r)^{-N} \times \sum_{\ell=0}^{N} \binom{N}{\ell} \left( \frac{p_{\rm rn}}{1 + {\rm ret_{up}}} \right)^{\ell} \left( \frac{1-p_{\rm rn}}{1 + {\rm ret_{down}}} \right)^{N-\ell} \\ &=\; (1+r)^{-N} \times \left\{ \frac{p_{\rm rn}}{1 + {\rm ret_{up}}} + \frac{1-p_{\rm rn}}{1 + {\rm ret_{down}}} \right\}^{N} \end{aligned}$$

Dabei haben wir in der letzten Zeile die allgemeine Binomische Formel

$$(a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} \, a^k \, b^{n-k}$$

benutzt.

**2. Aufgabe:** Wir benutzen wieder die Formel aus dem Theorem 5.2,

$$\begin{aligned} V_0 \;&=\; (1+r)^{-N} \times \sum_{\ell=0}^{N} H(S_{N,\ell}) \times \binom{N}{\ell} \, p_{\rm rn}^{\ell} \, (1-p_{\rm rn})^{N-\ell} \\ &\overset{r=0}{=}\; \sum_{\ell=0}^{N} H(S_{N,\ell}) \times \binom{N}{\ell} \, p_{\rm rn}^{\ell} \, (1-p_{\rm rn})^{N-\ell} \end{aligned}$$

mit der risikoneutralen W'keit

$$
\begin{aligned}
p_{\text{rn}} &= \frac{r \,-\, \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \,-\, \text{ret}_{\text{down}}} \\
&\overset{r=0}{\equiv} \frac{-\,\text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \,-\, \text{ret}_{\text{down}}} \\
&= \frac{-\,(-q)}{q \,-\, (-q)} \;=\; \frac{1}{2}
\end{aligned}
$$

und den Underlyingpreisen

$$
\begin{aligned}
S_{N,\ell} &= S_0\,(1 \,+\, \text{ret}_{\text{up}})^{\ell}\,(1 \,+\, \text{ret}_{\text{down}})^{N-\ell} \\
&= S_0\,(1+q)^{\ell}\,(1-q)^{N-\ell}
\end{aligned}
$$

Also,

$$
H(S_{N,\ell}) \;=\; S_{N,\ell}^2 \;=\; S_0^2 \times (1+q)^{2\ell}\,(1-q)^{2(N-\ell)}
$$

und wir bekommen

$$
\begin{aligned}
V_0 &= \sum_{\ell=0}^{N} H(S_{N,\ell}) \times \binom{N}{\ell} \left(\frac{1}{2}\right)^{\ell} \left(1-\frac{1}{2}\right)^{N-\ell} \\
&= S_0^2 \times \frac{1}{2^N} \sum_{\ell=0}^{N} \binom{N}{\ell} \left[(1+q)^2\right]^{\ell} \left[(1-q)^2\right]^{N-\ell} \\
&= S_0^2 \times \frac{1}{2^N} \times \left\{ (1+q)^2 \,+\, (1-q)^2 \right\}^N \\
&= S_0^2 \times \frac{1}{2^N} \times \left\{ 1+2q+q^2 \,+\, 1-2q+q^2 \right\}^N \\
&= S_0^2 \times \frac{1}{2^N} \times \left\{ 2 \,+\, 2q^2 \right\}^N \\
&= S_0^2 \times \left\{ 1 \,+\, q^2 \right\}^N .
\end{aligned}
$$