Hochschule RheinMain
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Lösungen zum 5. Übungsblatt
Einführung in die Finanzmathematik

1. Aufgabe: Wir betrachten ein N-Perioden Binomialmodell mit Preisprozess

$$S_k \;=\; S_{k-1} \times \begin{cases} (1+q) & \text{mit W'keit } \; p_{\text{up}} = 1/2 \\ (1-q) & \text{mit W'keit } \; p_{\text{down}} = 1/2 \end{cases}$$

für $k = 1, 2, ..., N$ mit $q \in (0,1)$, etwa $N = 250$ und $q = 1\%$. Berechnen Sie folgende Erwartungswerte:

a) $\mathsf{E}[\, S_N \,]$

b) $\mathsf{E}\big[\, S_N \,\big|\, \{S_j\}_{j=0}^k \,\big]$

c) $\mathsf{E}\big[\, \frac{1}{N} \sum_{m=1}^N S_m \,\big]$

d) $\mathsf{E}\big[\, \frac{1}{N} \sum_{m=1}^N S_m \,\big|\, \{S_j\}_{j=0}^k \,\big]$

e) $\mathsf{E}[\, \frac{S_0}{S_N} \,]$

f) $\mathsf{E}[\, \frac{S_0}{S_N} \,\big|\, \{S_j\}_{j=0}^k \,]$

Bemerkung: Die eigentliche Rechnung ist nicht so schwierig, die Übung besteht mehr darin, sich zu überlegen, ob das Ergebnis eine reine Zahl ist oder ob und welche Buchstaben dann noch im Ergebnis auftauchen.

Lösung: **a)** From the definition of the returns

$$\text{ret}_k \;:=\; \frac{S_k - S_{k-1}}{S_{k-1}}$$

we get

$$S_N \;=\; S_{N-1}(1 + \text{ret}_N)$$

and by induction

$$S_N \;=\; S_0 \prod_{k=1}^N \big(1 + \text{ret}_k\big)$$

Since all the returns are independent,

$$\begin{aligned} \mathsf{E}[\, S_N \,] \;&=\; \mathsf{E}\Big[S_0 \prod_{k=1}^N \big(1 + \text{ret}_k\big) \Big] \\ &=\; S_0 \prod_{k=1}^N \big(1 + \mathsf{E}[\text{ret}_k]\big) \,. \end{aligned}$$

Furthermore,

$$\begin{aligned}\mathsf{E}[\mathrm{ret}_k] &= +q \times p_{\mathrm{up}} \; + \; (-q) \times p_{\mathrm{down}} \\ &= q/2 - q/2 \; = \; 0\end{aligned}$$

Thus,

$$\mathsf{E}[\, S_N \,] \;=\; S_0 \prod_{k=1}^{N} \big(1+0\big) \;=\; S_0 \; .$$

b) The notation $\mathsf{E}\big[\cdots \,\big|\, \{S_j\}_{j=0}^k\,\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned}S_N &= S_0 \prod_{j=1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_0 \prod_{j=1}^{k} \big(1+\mathrm{ret}_j\big) \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big)\end{aligned}$$

and obtain

$$\begin{aligned}\mathsf{E}\big[\, S_N \,\big|\, \{S_j\}_{j=0}^k\,\big] &= \mathsf{E}\Big[\, S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \,\big|\, \{S_j\}_{j=0}^k\,\Big] \\ &= S_k \prod_{j=k+1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_j]\big) \\ &= S_k \prod_{j=k+1}^{N} \big(1+0\big) \;=\; S_k \; .\end{aligned}$$

c) Because of part (a), we have

$$\begin{aligned}\mathsf{E}\big[\,\frac{1}{N}\sum_{m=1}^{N} S_m\,\big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big] \\ &= \frac{1}{N}\sum_{m=1}^{N} S_0 \;=\; S_0 \; .\end{aligned}$$

d) Because of part (b), we obtain

$$\begin{aligned}\mathsf{E}\Big[\,\frac{1}{N}\sum_{m=1}^{N} S_m \,\big|\, \{S_j\}_{j=0}^k\,\Big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k\,\big] \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k\,\big] \;+\; \sum_{m=k+1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k\,\big]\Big\} \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k} S_m \;+\; \sum_{m=k+1}^{N} S_k\Big\} \\ &= \frac{k}{N} \times \frac{1}{k}\sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N} \times S_k \; .\end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N = S_0 \prod_{k=1}^{N} \big(1 + \text{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} = \prod_{k=1}^{N} \frac{1}{1 + \text{ret}_k} .$$

Since all the returns are independent,

$$\begin{aligned}
\mathsf{E}[\, S_0/S_N \,] &= \mathsf{E}\left[\prod_{k=1}^{N} \frac{1}{1+\text{ret}_k}\right] \\
&= \prod_{k=1}^{N} \mathsf{E}\left[\frac{1}{1+\text{ret}_k}\right] \\
&= \prod_{k=1}^{N} \left\{\frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2}\right\} \\
&= \prod_{k=1}^{N} \left\{\frac{1}{1-q^2}\right\} \\
&= \frac{1}{(1-q^2)^N} .
\end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots \big| \{S_j\}_{j=0}^k\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\text{ret}_{k+1}, \text{ret}_{k+2}, ..., \text{ret}_N$. Therefore we write as in part (b)

$$S_0 \,/\, S_N = S_0 \,/\, \Big\{S_k \prod_{j=k+1}^{N} \big(1 + \text{ret}_j\big)\Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}[\, S_0/S_N \,\big|\, \{S_j\}_{j=0}^k \,] &= S_0/S_k \; \mathsf{E}\left[\prod_{m=k+1}^{N} \frac{1}{1+\text{ret}_m} \,\Big|\, \{S_j\}_{j=0}^k\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\left[\frac{1}{1+\text{ret}_m}\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2}\right\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1-q^2}\right\} \\
&= S_0/S_k \, \frac{1}{(1-q^2)^{N-k}} . \qquad \blacksquare
\end{aligned}$$