

Lösungen zum 12. Übungsblatt
Einführung in die Finanzmathematik

Aufgabe 1: Using the Black-Scholes formulae, we get

$$\begin{aligned}d_{\pm} &= \frac{\log[S_0/K] + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\log[100/120] + (0.03 \pm (0.25)^2/2)4}{0.25 \times \sqrt{4}}\end{aligned}$$

which gives

$$\begin{aligned}d_+ &= 0.125357 \\ d_- &= -0.37464\end{aligned}$$

In the exam, you will get a table with $N(x)$ -values from which you can read off the numbers (probably with less digits, then giving a more unprecise result)

$$\begin{aligned}N(d_+) &= 0.549879 \\ N(d_-) &= 0.353963\end{aligned}$$

Thus,

$$\begin{aligned}V_0 &= S_0N(d_+) - Ke^{-rT}N(d_-) \\ &= 100 \times 0.549879 - 120e^{-0.12} \times 0.353963 = 17.3155.\end{aligned}$$

Aufgabe 2: Nach den Black-Scholes Formeln ist

$$\begin{aligned}V_{\text{Call},0}^{\text{BS}} &= S_0N(d_+) - Ke^{-rT}N(d_-) \\ V_{\text{Put},0}^{\text{BS}} &= -S_0N(-d_+) + Ke^{-rT}N(-d_-)\end{aligned}$$

so dass

$$\begin{aligned}V_{\text{Call},0}^{\text{BS}} - V_{\text{Put},0}^{\text{BS}} &= S_0 \underbrace{[N(d_+) + N(-d_+)]}_{=1} - Ke^{-rT} \underbrace{[N(d_-) + N(-d_-)]}_{=1} \\ &= S_0 - Ke^{-rT}\end{aligned}$$

da wegen

$$\begin{aligned}
 N(-x) &= \int_{-\infty}^{-x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
 &= \int_{-\infty}^{-x} e^{-\frac{(-y)^2}{2}} (-1) \frac{d(-y)}{\sqrt{2\pi}} \\
 &\stackrel{v:=-y}{=} \int_{+\infty}^{+x} e^{-\frac{v^2}{2}} (-1) \frac{dv}{\sqrt{2\pi}} \\
 &= \int_x^{\infty} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}}
 \end{aligned}$$

die Gleichung

$$\begin{aligned}
 N(x) + N(-x) &= \int_{-\infty}^x e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} + \int_{-\infty}^{-x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
 &= \int_{-\infty}^x e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} + \int_x^{\infty} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\
 &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = 1
 \end{aligned}$$

erfüllt ist.

Aufgabe 3: a) We have

$$\begin{aligned}
 H_{\text{call,perf}}(S_T) &= \max\{S_T/S_0 - 1, 0\} \\
 &= \frac{1}{S_0} \max\{S_T - S_0, 0\} \\
 &=: \frac{1}{S_0} H_{\text{call,abs}}(S_T)
 \end{aligned}$$

where $H_{\text{call,abs}}(S_T) = H_{\text{call}}(S_T)$ is just a standard call in absolute amount to which the Black-Scholes formula can be applied. Thus, since $K = S_0$ and $r = 0$,

$$\begin{aligned}
 V_0 &= \text{price}(H_{\text{call,perf}}) \\
 &= \frac{1}{S_0} \text{price}(H_{\text{call,abs}}) \\
 &= \frac{1}{S_0} \{S_0 N(d_+) - S_0 N(d_-)\} \\
 &= N(d_+) - N(d_-)
 \end{aligned}$$

with

$$\begin{aligned}
 d_{\pm} &= \frac{\log\left[\frac{S_0}{S_0}\right] + (0 \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\
 &= \pm \frac{\sigma\sqrt{T}}{2}.
 \end{aligned}$$

This proves part (a).

b) Using (a), we can write

$$\begin{aligned}V_0 &= N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2) \\&= \frac{N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2)}{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)} \times \{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)\} \\&\approx N'(0) \times \sigma\sqrt{T} \\&= \frac{1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} \times \sigma\sqrt{T} \\&= \frac{1}{\sqrt{2\pi}} \times \sigma\sqrt{T} \\&= 0.398942.. \times \sigma\sqrt{T} \\&\approx 0.4 \times \sigma\sqrt{T} .\end{aligned}$$