Hochschule RheinMain
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Lösungen zum 11. Übungsblatt
Einführung in die Finanzmathematik

Aufgabe 1: Wir haben

$$\begin{aligned} V_0 &= e^{-rT} \int_{\mathbb{R}} H(S_T)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

Wegen

$$\begin{aligned} & S_T \geq K \\ \Leftrightarrow\quad & S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T} \geq K \\ \Leftrightarrow\quad & \sigma\sqrt{T}\,x + (r-\sigma^2/2)T \geq \log[K/S_0] \\ \Leftrightarrow\quad & \sigma\sqrt{T}\,x \geq \log[K/S_0] - (r-\sigma^2/2)T \\ \Leftrightarrow\quad & x \geq \frac{\log[K/S_0] - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\ \Leftrightarrow\quad & x \geq -\frac{\log[S_0/K] + (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\ \Leftrightarrow\quad & x \geq -d_- \end{aligned}$$

bekommen wir

$$\begin{aligned} V_0 &= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{-d_-}^{\infty} 1\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT} \int_{-\infty}^{d_-} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-rT}\, N(d_-)\,. \end{aligned}$$

Aufgabe 2: a) Wir haben

$$\frac{p_{\text{rn}}}{1+\text{ret}_{\text{up}}} + \frac{1-p_{\text{rn}}}{1+\text{ret}_{\text{down}}} = \frac{p_{\text{rn}}(1+\text{ret}_{\text{down}}) \;+\; (1-p_{\text{rn}})(1+\text{ret}_{\text{up}})}{(1+\text{ret}_{\text{up}})(1+\text{ret}_{\text{down}})}$$

$$= \frac{1}{2} \times \frac{(1+\alpha\sqrt{\Delta t})(1+\mu\Delta t - \sigma\sqrt{\Delta t}) \;+\; (1-\alpha\sqrt{\Delta t})(1+\mu\Delta t + \sigma\sqrt{\Delta t})}{\big[(1+\mu\Delta t)+\sigma\sqrt{\Delta t})\big]\big[(1+\mu\Delta t)-\sigma\sqrt{\Delta t})\big]}$$

$$= \frac{1}{2} \times \frac{2(1+\mu\Delta t) \;-\; 2\alpha\sqrt{\Delta t}\sigma\sqrt{\Delta t}}{(1+\mu\Delta t)^2 \;-\; \sigma^2\Delta t}$$

$$= \frac{1+\mu\Delta t \;-\; \alpha\sigma\Delta t}{1+2\mu\Delta t+\mu^2\Delta t^2 \;-\; \sigma^2\Delta t} = \frac{1+(2\mu-r)\Delta t}{1+(2\mu-\sigma^2)\Delta t+\mu^2\Delta t^2} \;.$$

Mit $N = T/\Delta T$ folgt dann die angegebene Formel.

b) Wir ersetzen die Δt durch T/N und bekommen

$$\begin{aligned}
V_0^{\text{Bin}} &= e^{-rT}\left\{\frac{1+(2\mu-r)\frac{T}{N}}{1+(2\mu-\sigma^2)\frac{T}{N}+\mu^2\frac{T^2}{N^2}}\right\}^N \\
&= e^{-rT}\left\{\frac{1+(2\mu-\sigma^2)\frac{T}{N}+\mu^2\frac{T^2}{N^2}+(\sigma^2-r)\frac{T}{N}-\mu^2\frac{T^2}{N^2}}{1+(2\mu-\sigma^2)\frac{T}{N}+\mu^2\frac{T^2}{N^2}}\right\}^N \\
&= e^{-rT}\left\{1 \;+\; \frac{(\sigma^2-r)\frac{T}{N}-\mu^2\frac{T^2}{N^2}}{1+(2\mu-\sigma^2)\frac{T}{N}+\mu^2\frac{T^2}{N^2}}\right\}^N \\
&= e^{-rT}\left\{1 \;+\; \frac{(\sigma^2-r)T}{N} \;+\; O\big(\tfrac{1}{N^2}\big)\right\}^N \\
&\overset{N\to\infty}{\to} e^{-rT}\, e^{(\sigma^2-r)T} \;=\; e^{(\sigma^2-2r)T}
\end{aligned}$$

c) Wir bekommen:

$$\begin{aligned}
V_0^{\text{BS}} &= e^{-rT}\int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x+(r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT}\int_{\mathbb{R}} \frac{S_0}{S_0\, e^{\sigma\sqrt{T}\,x+(r-\sigma^2/2)T}}\, e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT}\int_{\mathbb{R}} e^{-\sigma\sqrt{T}\,x-(r-\sigma^2/2)T}\, e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT}e^{-(r-\sigma^2/2)T}\int_{\mathbb{R}} e^{-\sigma\sqrt{T}\,x}\, e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-2rT}e^{\sigma^2/2T}\int_{\mathbb{R}} e^{-\sigma\sqrt{T}\,x}\, e^{-\frac{x^2}{2}}\, e^{-\frac{\sigma^2 T}{2}}\,\frac{dx}{\sqrt{2\pi}}\; e^{\frac{\sigma^2 T}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= e^{(-2r+\sigma^2)T} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} \frac{dx}{\sqrt{2\pi}} \\
&= e^{(-2r+\sigma^2)T} \times 1 \\
&= e^{(\sigma^2-2r)T} .
\end{aligned}
$$