week9b: Das zeitabhängige Black-Scholes Modell und Kalibrierung an Marktpreise, Teil4

Wir muessen uns zunächst noch den Teil (b) vom Theorem 13.5 anschauen und dann das Korollar 13.6 beweisen, das sind dann die Black-Scholes Formeln mit zeitabhängigen Volatilitäten:

Theorem 13.5: a) Let $\{x_t\}_{t\geq 0}$ be a Brownian motion, let W be the Wiener measure and let σ_t be some deterministic function. Let F be some function. Then

$$\begin{aligned} \mathsf{E}_W\Big[\,F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,\Big] &= \int F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,dW(\{x_t\}_{0<t\leq T}) \\ &= \int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx \end{aligned} \tag{1}$$

where the implied volatility $\sigma_{\mathrm{imp},T}$ is given by

$$\sigma^2_{\mathrm{imp},T} \;:=\; \tfrac{1}{T}\textstyle\int_0^T \sigma_t^2\,dt \tag{2}$$

b) Let $H = H(S_T)$ be the payoff of some non path-dependent option. Then its fair value $V_0 = V_0^{\mathrm{BSTD}}$ in the time-dependent Black-Scholes model is given by

$$V_0^{\mathrm{BSTD}} \;=\; e^{-rT}\int_{\mathbb{R}} H\Big(S_0\,e^{\sigma_{\mathrm{imp},T}\sqrt{T}\,x\,+\,(r-\frac{\sigma^2_{\mathrm{imp},T}}{2})T}\Big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx \tag{3}$$

with an implied volatility $\sigma_{\mathrm{imp},T}$ is given by (2).

Proof: For part (b), we use Theorem 13.4 to obtain

$$\begin{aligned} V_0 &= e^{-rT}\mathsf{E}_W\big[H(S_T^{(r)})\big] \\ &= e^{-rT}\int H\big(S_0\,e^{\int_0^T \sigma_t\,dx_t + \int_0^T (r-\frac{\sigma_t^2}{2})dt}\big)\,dW(\{x_t\}_{0<t\leq T}) \end{aligned} \tag{4}$$

such that part (b) is an immediate consequence of (1) of part (a). ■

As an immediate consequence of Theorem 13.5, we are in a position to write down the Black-Scholes formulae for the fair value of call- and put-options in the time dependent Black-Scholes model. Da kann man dann natürlich auch immer leicht eine Klausuraufgabe zu machen, das könnte dann so aussehen wie in der 2. Aufgabe auf dem neuen Übungsblatt 8.

Corrolary 13.6: Consider standard european call- and put-options with strike K and maturity T,

$$\begin{aligned} H_{\text{call}}(S_T) &= \max\{S_T - K, 0\} \\ H_{\text{put}}(S_T) &= \max\{K - S_T, 0\} \end{aligned}$$

Suppose that the underlying real world asset price dynamics is given by the time-dependent Black-Scholes model

$$dS_t/S_t = \mu_t\, dt + \sigma_t\, dx_t \tag{5}$$

with some deterministic drift function μ_t and volatility function σ_t and let r be a (constant) interest rate. Define the implied volatility $\sigma_{\text{imp},T}$ at maturity T through the formula

$$\sigma_{\text{imp},T} = \left\{ \tfrac{1}{T} \int_0^T \sigma_t^2\, dt \right\}^{1/2} \tag{6}$$

Then the time zero fair values of calls and puts are given by (BSTD for <u>B</u>lack-<u>S</u>choles <u>T</u>ime <u>D</u>ependent)

$$V_{\text{call},0}^{\text{BSTD}} = S_0\, N(d_+) \,-\, K\, e^{-rT}\, N(d_-) \tag{7}$$

$$V_{\text{put},0}^{\text{BSTD}} = -S_0\, N(-d_+) \,+\, K\, e^{-rT}\, N(-d_-) \tag{8}$$

where

$$d_\pm := \frac{\log \frac{S_0}{K} + (r \pm \frac{\sigma_{\text{imp},T}^2}{2})T}{\sigma_{\text{imp},T}\sqrt{T}} \tag{9}$$

Proof: Follows immediately from (3) of Theorem 13.5 and the calculations in the proof of Theorem 6.1 of FM1. ■

Calibration to Market

Let $V_{\text{call},K,T}^{\text{market}}$ be an observed market price for a call option with strike K and maturity T. Let $V_{\text{call},K,T}^{\text{BS}}(\sigma)$ be the theoretical fair value when this option is priced in the time-independent Black-Scholes model with a constant volatility σ. Then the implied volatility σ_{imp} of this option is defined to be that constant volatility number which has to be put into the time-independent Black-Scholes model in order to reproduce the market price. That is,

$$V_{\text{call},K,T}^{\text{BS}}\big(\sigma_{\text{imp}}\big) \stackrel{!}{=} V_{\text{call},K,T}^{\text{market}} \tag{10}$$

If one looks at concrete market prices, one finds that σ_{imp} is in fact a function of K and T,

$$\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T) \tag{11}$$

With a time-dependent Black-Scholes model, we can take care of the T-dependence of the volatilities, but not of the K-dependence. If we denote by $V_{\text{call},K,T}^{\text{BSTD}}$ the theoretical fair value of a standard call priced in the time dependent Black-Scholes model with a time dependent volatility $\{\sigma_t\}_{0\leq t\leq T}$, then we obtain from Corollary 13.6

$$V^{\text{BS}}_{\text{call},K,T}\big(\sigma_{\text{imp},T}\big) \;=\; V^{\text{BSTD}}_{\text{call},K,T}\big(\{\sigma_t\}_{0\leq t\leq T}\big) \tag{12}$$

if the implied volatility for maturity T and the volatility function $\{\sigma_t\}_{0\leq t\leq T}$ are related through the equation

$$T\,\sigma^2_{\text{imp},T} \;=\; \int_0^T \sigma_t^2\,dt\;. \tag{13}$$

Now let

$$V^{\text{market}}_{\text{call},K,T_1},\; V^{\text{market}}_{\text{call},K,T_2},\;\cdots\;,\; V^{\text{market}}_{\text{call},K,T_m} \tag{14}$$

be a set of observed market prices (say, for strike $K = S_0$, for 'at the money' calls) and let

$$\sigma_{\text{imp},T_1},\;\sigma_{\text{imp},T_2},\;\cdots\;,\;\sigma_{\text{imp},T_m} \tag{15}$$

be the corresponding implied volatilities. Now, calibrating the time-dependent Black-Scholes model to these market quotes means that we have to determine the volatility function σ_t of the time-dependent Black-Scholes model in such a way that the equation

$$T_k\,\sigma^2_{\text{imp},T_k} \;=\; \int_0^{T_k} \sigma_t^2\,dt \tag{16}$$

is fulfilled for all observed maturities $T_1,\cdots,T_m$. From (16) we get

$$T_k\,\sigma^2_{\text{imp},T_k} - T_{k-1}\,\sigma^2_{\text{imp},T_{k-1}} \;=\; \int_{T_{k-1}}^{T_k} \sigma_t^2\,dt \tag{17}$$

Thus, if we let σ_t be a piecewise constant function, being equal to a constant σ_k on the intervalls (T_{k-1},T_k), then we get from (17)

$$\sigma_k^2 \;=\; \frac{T_k\,\sigma^2_{\text{imp},T_k} - T_{k-1}\,\sigma^2_{\text{imp},T_{k-1}}}{T_k - T_{k-1}} \tag{18}$$

The process of choosing the σ_k according to (18) when the σ_{imp,T_k} are given by market quotes is called 'calibrating the time-dependent Black-Scholes model to the market'.

Und zu dieser Formel (18) gibt es auch immer eine Standardaufgabe in der Klausur, das würde dann etwa so aussehen wie in der 3. Aufgabe auf dem neuen Übungsblatt 8.