week9a: Das zeitabhängige Black-Scholes Modell und Kalibrierung an Marktpreise, Teil3

Nachdem wir letzte Woche die allgemeine Pricing-Formel für pfadabhängige oder pfadunabhängige Optionen im zeitabhängigen Black-Scholes Modell bewiesen haben,

$$V_0 \;=\; e^{-rT}\mathsf{E}_W\big[H(S_{t_1}^{(r)},...,S_{t_m}^{(r)})\big] \tag{1}$$

mit $\mathsf{E}_W[\,\cdot\,]$ der Erwartungswert bezüglich des Standard Wiener-Maßes und $S_t^{(r)}$ der risikoneutrale Preisprozess, mit dem r, nicht mit dem μ_t, gegeben durch

$$S_t^{(r)} \;=\; S_0\, e^{\int_0^t \sigma_u\, dx_u \,+\, \int_0^t (r-\frac{\sigma_u^2}{2})du} \tag{2}$$

oder äquivalent

$$dS_t^{(r)} \,/\, S_t^{(r)} \;=\; r\,dt \;+\; \sigma_t\, dx_t \tag{3}$$

das war eine recht theoretische Sache, können wir jetzt das folgende, sehr klausurrelevante Theorem 13.5 formulieren und beweisen. Die Numerierung ist wieder die aus dem Gesamt-Skript `http://hsrm-mathematik.de/WS201516/master/option-pricing/Chapter13.pdf`.

Theorem 13.5: a) Let $\{x_t\}_{t\geq 0}$ be a Brownian motion, let W be the Wiener measure and let σ_t be some deterministic function. Let F be some function. Then

$$\begin{aligned}\mathsf{E}_W\Big[\,F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,\Big] \;&=\; \displaystyle\int F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,dW(\{x_t\}_{0<t\leq T}) \\ &=\; \displaystyle\int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx\end{aligned} \tag{4}$$

where the implied volatility $\sigma_{\mathrm{imp},T}$ is given by

$$\sigma_{\mathrm{imp},T}^2 \;:=\; \tfrac{1}{T}\int_0^T \sigma_t^2\,dt \tag{5}$$

b) Let $H = H(S_T)$ be the payoff of some non path-dependent option. Then its fair value V_0 in the time-dependent Black-Scholes model is given by

$$V_0 \;=\; e^{-rT}\int_{\mathbb{R}} H\Big(S_0\, e^{\sigma_{\mathrm{imp},T}\sqrt{T}\,x \,+\, (r-\frac{\sigma_{\mathrm{imp},T}^2}{2})T}\Big)\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx \tag{6}$$

with an implied volatility $\sigma_{\mathrm{imp},T}$ is given by (5).

Proof: a) Der Beweis ist sehr ähnlich zu den Rechnungen, die wir im Kapitel 4 in der FM1 gemacht haben, und zwar insbesondere analog zu den Formeln zur Variablensubstitution in

(4.11) und dann brauchen wir das Lemma 4.1 mit der Formel (4.15), das wird gleich benutzt:

$$\begin{aligned}
\mathsf{E}_W\Big[\,F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\Big] &= \int F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\,dW(\{x_t\}_{0<t\le T}) \\
&= \lim_{\Delta t\to 0}\int F\Big(\sqrt{\Delta t}\sum_{j=1}^{N_T}\sigma_{t_j}\,\phi_j\Big)\prod_{j=1}^{N_T} e^{-\frac{\phi_j^2}{2}}\,\frac{d\phi_j}{\sqrt{2\pi}} \\
&= \lim_{\Delta t\to 0}\int F\Big(\sqrt{\Delta t}\sum_{j=1}^{N_T}\phi_j\Big)\prod_{j=1}^{N_T} e^{-\frac{\phi_j^2}{2\sigma_{t_j}^2}}\,\frac{d\phi_j}{\sqrt{2\pi\sigma_{t_j}^2}} \\
&\overset{(4.11)}{=} \lim_{\Delta t\to 0}\int F(y_T)\prod_{j=1}^{N_T} e^{-\frac{(y_{t_j}-y_{t_{j-1}})^2}{2\Delta t\sigma_{t_j}^2}}\,\frac{dy_{t_j}}{\sqrt{2\pi\Delta t\sigma_{t_j}^2}}
\end{aligned}\tag{7}$$

Because of (4.15)

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\Delta t\sigma_{t_j}^2}}\,e^{-\frac{(y_{t_j}-y_{t_{j-1}})^2}{2\Delta t\sigma_{t_j}^2}}\;\frac{1}{\sqrt{2\pi\Delta t\sigma_{t_{j+1}}^2}}\,e^{-\frac{(y_{t_{j+1}}-y_{t_j})^2}{2\Delta t\sigma_{t_{j+1}}^2}}\;dy_{t_j} \\
&\quad= \int_{\mathbb{R}} p_{\Delta t\sigma_{t_j}^2}(y_{t_{j-1}},y_{t_j})\,p_{\Delta t\sigma_{t_{j+1}}^2}(y_{t_j},y_{t_{j+1}})\,dy_{t_j} \\
&\quad= p_{\Delta t\sigma_{t_j}^2+\Delta t\sigma_{t_{j+1}}^2}(y_{t_{j-1}},y_{t_{j+1}}) \\
&\quad= \frac{1}{\sqrt{2\pi\Delta t(\sigma_{t_j}^2+\sigma_{t_{j+1}}^2)}}\,e^{-\frac{(y_{t_{j+1}}-y_{t_{j-1}})^2}{2\Delta t(\sigma_{t_j}^2+\sigma_{t_{j+1}}^2)}}
\end{aligned}\tag{8}$$

Thus, in (7) we can integrate out all y_{t_j} with the exception of y_T to obtain ($s^2 := \sum_{j=1}^{N_T}\sigma_{t_j}^2$)

$$\begin{aligned}
\mathsf{E}_W\Big[\,F\big(\textstyle\int_0^T \sigma_t\,dx_t\big)\Big] &= \lim_{\Delta t\to 0}\int_{\mathbb{R}} F(y_T)\,\frac{1}{\sqrt{2\pi\Delta t\,s^2}}\,e^{-\frac{(y_T-y_{t_0})^2}{2\Delta t\,s^2}}\,dy_T \\
&= \int_{\mathbb{R}} F(y_T)\,\frac{1}{\sqrt{2\pi T\sigma_{\mathrm{imp},T}^2}}\,e^{-\frac{y_T^2}{2T\sigma_{\mathrm{imp},T}^2}}\,dy_T \\
&= \int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx
\end{aligned}\tag{9}$$

since

$$\Delta t\,s^2 \;=\; \Delta t\textstyle\sum_{j=1}^{N_T}\sigma_{t_j}^2 \;\overset{\Delta t\to 0}{\to}\; \int_0^T\sigma_t^2\,dt \;=\; T\,\sigma_{\mathrm{imp},T}^2$$

This proves part (a). For part (b), we use formulae (13.33) and (13.34) of Theorem 13.4 to obtain

$$\begin{aligned}
V_0 &= e^{-rT}\,\mathsf{E}_W\big[H(S_T^{(r)})\big] \\
&= e^{-rT}\int H\big(\,S_0\,e^{\int_0^T\sigma_t\,dx_t+\int_0^T(r-\frac{\sigma_t^2}{2})dt}\,\big)\,dW(\{x_t\}_{0<t\le T})
\end{aligned}\tag{10}$$

such that part (b) is an immediate consequence of (4) of part (a). ■

As an immediate consequence of Theorem 13.5, we are in a position to write down the Black-Scholes formulae for the fair value of call- and put-options in the time dependent Black-Scholes model. Da kann man dann natürlich auch immer leicht eine Klausuraufgabe zu machen, das könnte dann so aussehen wie die 2. Aufgabe auf dem neuen Übungsblatt.

Corrolary 13.6: Consider standard european call- and put-options with strike K and maturity T,

$$\begin{aligned} H_{\text{call}}(S_T) &= \max\{S_T - K, 0\} \\ H_{\text{put}}(S_T) &= \max\{K - S_T, 0\} \end{aligned}$$

Suppose that the underlying asset price dynamics is given by the time-dependent Black-Scholes model

$$dS_t/S_t = \mu_t \, dt + \sigma_t \, dx_t \tag{11}$$

with some deterministic drift function μ_t and volatility function σ_t. Define the implied volatility $\sigma_{\text{imp},T}$ for maturity T through the formula

$$\sigma_{\text{imp},T} = \left\{ \tfrac{1}{T} \int_0^T \sigma_t^2 \, dt \right\}^{1/2} \tag{12}$$

Then the time zero fair values of calls and puts are given by

$$V_{\text{call},0} = S_0 \, N(d_+) \, - \, K \, e^{-rT} \, N(d_-) \tag{13}$$

$$V_{\text{put},0} = -S_0 \, N(-d_+) \, + \, K \, e^{-rT} \, N(-d_-) \tag{14}$$

where

$$d_\pm := \frac{\log \frac{S_0}{K} + (r \pm \frac{\sigma_{\text{imp},T}^2}{2})T}{\sigma_{\text{imp},T}\sqrt{T}} \tag{15}$$

Proof: Follows immediately from (6) of Theorem 13.5 and the calculations in the proof of Theorem 6.1 of FM1. ■