week5a: Probabilities Involving the Minimum and the Maximum of a Brownian Motion, Teil3

Das Hauptresultat dieses Kapitels ist jetzt also das folgende

Theorem 3.5: Let dW be the Wiener measure, let $\{x_t\}_{0\leq t\leq T}$ be a Brownian motion and recall

$$N(x) = \int_{-\infty}^{x} \varphi(y)\,dy, \qquad \varphi(y) = \tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}} \tag{1}$$

Let a, b be positive real numbers, $a \geq 0$ and $b \geq 0$. Then we have the following probabilities:

a)

$$\mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a,\, x_T < a-b\big) \;=\; \mathsf{P}_W\big(x_T > a+b\big) \tag{2}$$

$$\mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a\big) \;=\; 2\,\mathsf{P}_W\big(x_T \geq a\big) \tag{3}$$

b)

$$\mathsf{P}_W\big(x_T \leq a,\, \max_{0\leq t\leq T} x_t \leq b\big) \;=\; \begin{cases} N\big(a/\sqrt{T}\big) + N\big((2b-a)/\sqrt{T}\big) - 1 & \text{if } a \leq b \\ 2N\big(b/\sqrt{T}\big) - 1 & \text{if } a > b \end{cases} \tag{4}$$

c) In particular, there is the density

$$\mathsf{P}_W\Big(x_T \in [a, a+da),\, \max_{0\leq t\leq T} x_t \in [b, b+db)\Big) \;=\; -\tfrac{2}{T}\,\varphi'\big(\tfrac{2b-a}{\sqrt{T}}\big)\,\chi(a<b)\,da\,db\,. \tag{5}$$

Proof: a) Define the stopping time

$$\tau_a(\{x_t\}) = \inf_{t\in(0,\infty)} \{t \,|\, x_t = a\} \tag{6}$$

Then

$$\big\{x_t \,|\, \max_{0\leq t\leq T} x_t \geq a\big\} \;=\; \big\{x_t \,|\, \tau_a(\{x_t\}) \leq T\big\} \tag{7}$$

For given x_t, define the at the level $x_{\tau_a} = a$ reflected path as $y_t = x_t$ for $t \leq \tau_a$ and $y_t = 2x_{\tau_a} - x_t = 2a - x_t$ for $t > \tau_a$. In particular, $\tau_a = \tau_a(\{x_t\}_{0\leq t\leq\tau_a}) = \tau_a(\{y_t\}_{0\leq t\leq\tau_a})$. Then

$$\begin{aligned}
\mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a,\, x_T < a-b\big) &= \mathsf{P}_W\big(\tau_a \leq T,\, x_T < a-b\big) \\
&= \int \chi\big(\tau_a(\{x_t\}_{0\leq t\leq\tau_a}) \leq T,\, x_T < a-b\big)\, dW(\{x_t\}) \\
&= \int \chi\big(\tau_a(\{y_t\}_{0\leq t\leq\tau_a}) \leq T,\, 2a - y_T < a-b\big)\, dW(\{y_t\}) \\
&= \int \chi\big(\tau_a(\{y_t\}_{0\leq t\leq\tau_a}) \leq T,\, y_T > a+b\big)\, dW(\{y_t\}) \\
&= \int \chi\big(y_T > a+b\big)\, dW(\{y_t\}) \\
&= \mathsf{P}_W\big(x_T > a+b\big) \qquad (8)
\end{aligned}$$

In the third line of (8) we used the reflection principle Lemma 3.4 whereas in the last line we simply renamed the integration variables from y_t to x_t. In particular, for $b = 0$,

$$\mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a,\, x_T < a\big) = \mathsf{P}_W\big(x_T > a\big) \qquad (9)$$

Using this, (3) follows from

$$\begin{aligned}
\mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a\big) &= \mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a, x_T < a\big) + \mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \geq a, x_T \geq a\big) \\
&\overset{(9)}{=} \mathsf{P}_W\big(x_T > a\big) + \mathsf{P}_W\big(x_T \geq a\big) \\
&= 2\,\mathsf{P}_W\big(x_T \geq a\big) \qquad (10)
\end{aligned}$$

b) For $b < a$ we have

$$\begin{aligned}
\mathsf{P}_W\big(x_T \leq a,\, \max_{0\leq t\leq T} x_t \leq b\big) &= \mathsf{P}_W\big(\max_{0\leq t\leq T} x_t \leq b\big) \\
&= 1 - \mathsf{P}_W\big(\max_{0\leq t\leq T} x_t > b\big) \\
&\overset{(a)}{=} 1 - 2\,\mathsf{P}_W\big(x_T > b\big) \\
&= 2\,\mathsf{P}_W\big(x_T \leq b\big) - 1 \\
&= 2\int_{\mathbb{R}} \chi(x_T \leq b)\, p_T(0, x_T)\, dx_T \;-\; 1 \\
&= 2\int_{-\infty}^{b/\sqrt{T}} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy \;-\; 1 \\
&= 2N\big(b/\sqrt{T}\big) - 1 \qquad (11)
\end{aligned}$$

For $a \le b$,

$$\begin{aligned}
\mathsf{P}_W\big(\,x_T \le a,\ \max_{0\le t\le T} x_t \le b\,\big) &= \mathsf{P}_W\big(\,x_T \le a\,\big) - \mathsf{P}_W\big(\,x_T \le a,\ \max_{0\le t\le T} x_t > b\,\big) \\
&= \mathsf{P}_W\big(\,x_T \le a\,\big) - \mathsf{P}_W\big(\,\max_{0\le t\le T} x_t > b,\ x_T \le b-(b-a)\,\big) \\
&\overset{(a)}{=} \mathsf{P}_W\big(\,x_T \le a\,\big) - \mathsf{P}_W\big(\,x_T > b+(b-a)\,\big) \\
&= \mathsf{P}_W\big(\,x_T \le a\,\big) + \mathsf{P}_W\big(\,x_T \le 2b-a\,\big) - 1 \\
&= N\big(a/\sqrt{T}\big) + N\big((2b-a)/\sqrt{T}\big) - 1 \qquad (12)
\end{aligned}$$

This proves (4). The density in (5) in part (c), say $\rho(a,b)$, is obtained by differentiation,

$$\rho(a,b) \;=\; \frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathsf{P}_W\big(\,x_T \le a,\ \max_{0\le t\le T} x_t \le b\,\big) \qquad (13)$$

One computes

$$\begin{aligned}
\rho(a,b) &= \frac{\partial}{\partial b}\begin{cases} \frac{1}{\sqrt{T}}\Big(\varphi\big(\frac{a}{\sqrt{T}}\big) - \varphi\big(\frac{2b-a}{\sqrt{T}}\big)\Big) & \text{if } a<b \\ 0 & \text{if } a>b \end{cases} \\
&= -\tfrac{2}{T}\,\varphi'\big(\tfrac{2b-a}{\sqrt{T}}\big)\,\chi(a<b) \qquad (14)
\end{aligned}$$

Since the first line of (14) is continuous at $a=b$, there is no δ-function at $a=b$. Finally observe that

$$\begin{aligned}
\mathsf{P}_W\big(\,x_T \ge -a,\ \min_{0\le t\le T} x_t \ge -b\,\big) &= \mathsf{P}_W\big(\,-x_T \le a,\ -\min_{0\le t\le T} x_t \le b\,\big) \\
&= \mathsf{P}_W\big(\,-x_T \le a,\ \max_{0\le t\le T}\{-x_t\} \le b\,\big) \\
&= \mathsf{P}_W\big(\,x_T \le a,\ \max_{0\le t\le T} x_t \le b\,\big)
\end{aligned}$$

since $dW(\{-x_t\}) = dW(\{x_t\})$. ■