Vorlesung 2: Auffrischung Finanzmathematik I, Teil2

Wir müssen uns an Brownsche Bewegung, Wiener-Maß und das Black-Scholes Modell erinnern. Das sind natürlich sehr wichtige Sachen, die wir dann auch gleich in der Finanzmathematik II benutzen werden, wenn wir etwa Preise von exotischen oder pfadabhängigen Optionen berechnen wollen wie Barrier-Optionen oder All-Time-High Optionen. In der FM1 hatten wir uns im 4. Kapitel damit befasst.

Brownian Motion, Wiener Measure and the Black-Scholes Model

Consider some discrete times t_k in the intervall $[0,T]$,

$$t_k = k\tfrac{T}{N} = k\Delta t\,, \quad k = 0, 1, ..., N \tag{1}$$

where

$$N = N_T = \tfrac{T}{\Delta t} \in \mathbb{N} \tag{2}$$

Let $S_{t_k} = S_{k\Delta t}$ be the price of some stock at time t_k and denote the returns by going from one time step to the next by

$$\mathrm{ret}_{t_k} \;=\; \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \tag{3}$$

One may think of Δt being one day and S_{t_k} being the closing prices at each day although later we will consider the limit $\Delta t \to 0$. It is an empirical fact that the daily returns of many assets are bell shaped, like a Gaussian distribution. It is also an empirical fact that the mean scales with Δt and the standard deviation scales with $\sqrt{\Delta t}$. Thus, one is led to

$$\mathrm{ret}_{t_k} \;=\; \mu\,\Delta t + \sigma\sqrt{\Delta t}\,\phi_k \tag{4}$$

where the ϕ_k are identically independent normally distributed random variables with mean zero and variance one,

$$\phi_k \in \mathcal{N}(0,1) \quad \text{i.i.d.} \tag{5}$$

Equation (4) defines the Black-Scholes model in discrete time. Asset prices are given by

$$S_t = S_0 \prod_{k=1}^{N_t} \Big(1 + \mu\,\Delta t + \sigma\sqrt{\Delta t}\,\phi_k\Big) \tag{6}$$

with $N_t = t/\Delta t$. For $\mu = 0$ one obtains, using the second order Taylor expansion $\log(1+x) = x - x^2/2 + O(x^3)$ in the third line,

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} \left(1 + \sigma\, \Delta t^{1/2} \phi_k\right) \\ &= S_0\, e^{\sum_{k=1}^{N_t} \log(1+\sigma\, \Delta t^{1/2}\phi_k)} \\ &= S_0\, e^{\sum_{k=1}^{N_t} \left(\sigma\, \Delta t^{1/2}\phi_k - \frac{1}{2}\sigma^2 \Delta t \phi_k^2 + O(\Delta t^{3/2})\right)} \\ &= S_0\, e^{\sigma\, \Delta t^{1/2} \sum_{k=1}^{N_t} \phi_k - \frac{\sigma^2}{2} \Delta t \sum_{k=1}^{N_t} \phi_k^2 + O(N_t \Delta t^{3/2} = \Delta t^{1/2})} \end{aligned} \tag{7}$$

We consider the expectation

$$\mathsf{E}\left[f\left(\Delta t^{1/2} \textstyle\sum_{k=1}^{N_t} \phi_k\right)\right] = \int_{\mathbb{R}^{N_t}} f\left(\Delta t^{1/2} \textstyle\sum_{k=1}^{N_t} \phi_k\right) \prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}}\, d\phi_k \tag{8}$$

where f is some function. We make a substitution of variables $(\phi_k)_{1 \le k \le N_t} \to (x_k)_{1 \le k \le N_t}$ defined as follows:

$$\begin{array}{rcl} x_1 &=& \sqrt{\Delta t}\, \phi_1 \\ x_2 &=& \sqrt{\Delta t}\, (\phi_1 + \phi_2) \\ x_3 &=& \sqrt{\Delta t}\, (\phi_1 + \phi_2 + \phi_3) \\ &\vdots& \\ x_{N_t} &=& \sqrt{\Delta t}\, (\phi_1 + \phi_2 + ... + \phi_{N_t}) \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} \phi_1 &=& x_1/\sqrt{\Delta t} \\ \phi_2 &=& (x_2 - x_1)/\sqrt{\Delta t} \\ \phi_3 &=& (x_3 - x_2)/\sqrt{\Delta t} \\ &\vdots& \\ \phi_{N_t} &=& (x_{N_t} - x_{N_t - 1})/\sqrt{\Delta t} \end{array} \tag{9}$$

and instead of labelling the x with $k \in \{1, 2, ..., N_t\}$, we label them with $k\Delta t$ which has the meaning of time. In particular, $N_t \Delta t = t$. So, rename $x_k \to x_{k\Delta t}$. The Jacobian of the transformation (9) is $\det \frac{\partial \phi}{\partial x} = 1/\sqrt{\Delta t}^{N_t}$. The expectation (8) becomes

$$\mathsf{E}\left[f\left(\Delta t^{1/2} \textstyle\sum_{k=1}^{N_t} \phi_k\right)\right] = \int_{\mathbb{R}^{N_t}} f(x_t) \prod_{k=1}^{N_t} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{10}$$

where we introduced the kernel

$$p_\tau(x, y) := \frac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \tag{11}$$

and used the definition

$$x_0 := 0 \tag{12}$$

The kernel (11) has the following basic property:

Lemma 4.1: *Let $p_t(x, y)$ be given by (11). Then*

$$\textstyle\int_{\mathbb{R}} p_s(x, y) p_t(y, z)\, dy = p_{s+t}(x, z)\ . \tag{13}$$

Using this lemma, (10) simplifies to

$$\begin{aligned} \mathsf{E}\left[f\left(\Delta t^{1/2} \textstyle\sum_{k=1}^{N_t} \phi_k\right)\right] &= \int_{\mathbb{R}} f(x_t) p_{N_t \Delta t}(x_0, x_t)\, dx_t \\ &= \int_{\mathbb{R}} f(x_t)\, \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x_t^2}{2t}}\, dx_t \end{aligned} \tag{14}$$

since $x_0 = 0$.

Definition 4.1: *Let $N_T = T/\Delta t$. The measure*

$$dW(\{x_t\}_{0<t\leq T}) := \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{15}$$

is called the Wiener measure and the family of random variables $\{x_t\}_{0<t\leq T}$ is called a Brownian motion. In terms of i.i.d. random variables $\phi_k \in \mathcal{N}(0,1)$,

$$x_t = \lim_{\Delta t\to 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \phi_k \tag{16}$$

and, in discrete time,

$$dW \;=\; dW\big(\{\phi_k\}_{1\leq k\leq N_T}\big) \;=\; \prod_{k=1}^{N_T} e^{-\frac{\phi_k^2}{2}} \frac{d\phi_k}{\sqrt{2\pi}} \;.$$

Integrals with respect to the Wiener measure are computed according to the following very important

Theorem 4.1: *Let $F:\mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$. Then*

$$\int F(x_{t_1},...,x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) = \int_{\mathbb{R}^m} F(x_{t_1},...,x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \tag{17}$$

Now we return to (7). We have

$$S_t \;=\; S_0\, e^{\sigma\sqrt{\Delta t}\sum_{k=1}^{N_t}\phi_k - \frac{\sigma^2}{2}\Delta t \sum_{k=1}^{N_t}\phi_k^2 + O(\sqrt{\Delta t})} \tag{18}$$

The first term in the exponent converges to a Brownian motion $x_t = \lim_{\Delta t\to 0}\sqrt{\Delta t}\sum_{k=1}^{N_t}\phi_k$ and the last term vanishes, but what about the second term? There was the following

Theorem: Let

$$I_{\Delta t} \;:=\; \Delta t \sum_{k=1}^{t/\Delta t} \phi_k^2$$

Then

a) For arbitrary Δt,

$$\mathsf{E}[I_{\Delta t}] \;=\; t$$

b)

$$\lim_{\Delta t\to 0} \mathsf{V}[I_{\Delta t}] \;=\; 0$$

c) For any $\varepsilon > 0$,

$$\lim_{\Delta t \to 0} \mathsf{Prob}\big[\, |I_{\Delta t} - t| \geq \varepsilon \,\big] \;=\; 0$$

Thus the quantity

$$\Delta t \sum_{k=1}^{t/\Delta t} \phi_k^2 \;\overset{\Delta t \to 0}{\to}\; t$$

becomes actually deterministic in the limit $\Delta t \to 0$. Substituting this in the exponent of (18), we arrive at

$$\begin{aligned} S_t &= S_0\, e^{\sigma \sqrt{\Delta t} \sum_{k=1}^{N_t} \phi_k - \frac{\sigma^2}{2} \Delta t \sum_{k=1}^{N_t} \phi_k^2} \\ &= S_0\, e^{\sigma x_t - \frac{\sigma^2}{2} t} \end{aligned} \tag{19}$$

If we reintroduce the μ which we put to 0 in the beginning, one obtains

$$S_t = S_0\, e^{\mu t + \sigma x_t - \frac{\sigma^2}{2} t} \tag{20}$$

We summarize our results: The statistics of financial data suggests the model

$$\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \;=\; \frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\, \Delta t + \sigma \sqrt{\Delta t}\, \phi_k \tag{21}$$

In view of (9), in particular, the right hand side thereof, and recalling the relabelling $x_k \to x_{k\Delta t}$, we may write this as

$$\frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\, \Delta t + \sigma\, (x_{t_k} - x_{t_{k-1}}) = \mu\, \Delta t + \sigma\, \Delta x_{t_k} \tag{22}$$

or, in the continuous time limit $\Delta t \to 0$,

$$\frac{dS}{S} \;=\; \mu\, dt + \sigma\, dx_t \tag{23}$$

where $\{x_t\}_{0<t\leq T}$ is a Brownian motion. Equation (23) is called the **Black-Scholes SDE** and the asset price model (20), which is a solution of the Black-Scholes SDE, is called the **Black-Scholes model**. The solution (20) is usually also refered to as a 'geometric Brownian motion'.