**Vorlesung 1: Auffrischung Finanzmathematik I, Teil1**

Also wir müssten uns zunächst an einige grundlegende Sachen aus der Finanzmathematik I erinnern. Da wäre zunächst mal die Standard-Formel aus dem Kapitel 1, mit der wir dann ja, durch eine geeignete Wahl der $\delta_k$, Optionsauszahlungen repliziert hatten. Das entsprechende Material aus dem Kapitel 1 war da folgendes:

## Chapter 1: Trading Strategies

Suppose that some stock $S$ can be traded on $N$ trading days

$$t_1, t_2, \cdots, t_k, \cdots, t_{N-1}, t_N \tag{1}$$

and suppose that

$$S_k \;:=\; S(t_k) \tag{2}$$

denotes the price of the stock on day $t_k$. Suppose start time, say, today, is $t_0$ and $S_0 := S(t_0)$ denotes the price of the stock at start time.

Suppose that at start time we have an initial cash amount of $V_0$. This amount lies on some bank account and interest rates will be paid on that amount. To fix notation, let us assume that we have a constant yearly interest rate of $r$ (so, for example, $r = 3\%$) and that the $t_k$ are expressed in year fraction (that is, if, say, $t_0 = 0$ is Jan 1st 2014 and $t_k$ is supposed to be Oct 1st, 2014, then approximately $t_k \approx 0.75$). Thus, if we simply put $V_0$ on a bank account and do nothing (that is, no buying or selling of stocks) then at time $t_k$ we have a cash amount $V_k$ given by

$$V_k \;=\; e^{r(t_k - t_0)} V_0 \tag{3}$$

Now, consider the following trading strategy:

- At start time $t_0$ buy a number of $\delta_0$ stocks.
- At time $t_1$, sell these $\delta_0$ stocks and buy a number of $\delta_1$ stocks. Or, more precisely, if $\delta_1 - \delta_0 \geq 0$, at time $t_1$ buy a number of $\delta_1 - \delta_0$ stocks, or, if $\delta_1 - \delta_0 < 0$, at time $t_1$ sell a number of $\delta_1 - \delta_0$ stocks. To put it in another way: At time $t_1$ readjust your stock position such that you hold a number of $\delta_1$ stocks at the end of day $t_1$.
- At time $t_2$, sell these $\delta_1$ stocks and buy a number of $\delta_2$ stocks such that your stock position is $\delta_2$ stocks at the end of day $t_2$.

$$\vdots$$

- At time $t_k$, sell $\delta_{k-1}$ stocks and buy a number of $\delta_k$ stocks such that your stock position is $\delta_k$ stocks at the end of day $t_k$.

$$\vdots \tag{4}$$

- At time $t_N$, sell $\delta_{N-1}$ stocks and buy no new stocks such that your stock position is closed at the end of day $t_N$.

Then this trading strategy has generated the following amount of money:

**Theorem 1.1:** Consider the trading strategy (4). Then at time $t_k$ this strategy has generated an amount $V_k$ given by the following expressions, $k = 1, 2, ..., N-1, N$:

**a)** If the interest rates are zero, $r = 0$, then

$$V_k \;=\; V_0 + \sum_{j=1}^{k} \delta_{j-1}(S_j - S_{j-1}) \tag{5}$$

**b)** If the interest rates are non zero, $r \neq 0$, then

$$\begin{aligned} V_k &= e^{r(t_k-t_0)}V_0 + \sum_{j=1}^{k} \delta_{j-1}\left(e^{r(t_k-t_j)}S_j - e^{r(t_k-t_{j-1})}S_{j-1}\right) \\ &= e^{r(t_k-t_0)}\Big\{V_0 + \sum_{j=1}^{k} \delta_{j-1}\left(e^{-r(t_j-t_0)}S_j - e^{-r(t_{j-1}-t_0)}S_{j-1}\right)\Big\} \end{aligned} \tag{6}$$

This can be written more compactly if we define the discounted quantities

$$v_j \;:=\; e^{-r(t_j-t_0)}V_j\,, \quad j \in \{0, 1, ..., N\} \tag{7}$$

$$s_j \;:=\; e^{-r(t_j-t_0)}S_j\,, \tag{8}$$

then (6) becomes

$$v_k \;=\; v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \tag{9}$$

Vielleicht machen Sie sich auch nochmal klar, zumindest für den Fall Zinsen $r = 0$, das ist dann die Formel (5) aus Theorem 1.1.a, wie man darauf gekommen ist.

Die nächste Sache war dann das Binomialmodell, wo wir also für eine beliebige Optionsauszahlung $H = H(S_0, S_1, \cdots, S_N)$ Deltas $\delta_0, \delta_1, \cdots, \delta_{N-1}$ angeben konnten, mit denen es dann also möglich war, den option payoff $H$ exakt zu replizieren. Das entsprechende Theorem war das Theorem 2.1 aus dem Kapitel 2. Wir geben es hier, weiten unten dann, vielleicht in einer etwas kompakteren Notation an.

## Chapter 2: The Binomial Model

Let $S$ be some tradable asset with prices

$$S_k = S(t_k), \quad k = 0, 1, 2, ... \tag{10}$$

and let

$$H = H(S_0, S_1, ..., S_{N-1}, S_N) \tag{11}$$

be some option payoff with start date $t_0$ and end date or maturity $t_N$. We want to replicate the option payoff (11) with a suitable trading strategy in the underlying $S$. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding $\delta_k$ assets at the end of day $t_k$ generates the amount

$$V_N = V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{12}$$

Each $\delta_k$ will be determined on the end of trading day $t_k$. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, $\delta_k$ can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k = \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \tag{13}$$

**Definition 2.1:** We say that an option payoff (11) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of $\delta_k$ of the form (13) and some initial amount $V_0$ such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) = V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{14}$$

The initial amount $V_0$ which is needed to set up the replicating strategy is called the theoretical fair value of $H$. Also $V_0$ ist der Optionspreis.

**Definition 2.2:** If the price process $S_k = S(t_k)$ of some tradable asset $S$ has the dynamics

$$S_k = S_{k-1}(1 + \mathrm{ret}_k) \quad \text{with} \quad \mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\} \; \forall \, k = 1, 2, ... \tag{15}$$

then we say that $S$ is given by the Binomial model.

**Remark:** Observe that in Definition 2.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability $p_{\mathrm{up}}$ such that an up-return $\mathrm{ret}_{\mathrm{up}}$ will occur and a probability $p_{\mathrm{down}} = 1 - p_{\mathrm{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value $V_0$ are actually independent of such probabilities. Nevertheless we have to remark that the definition of the Binomial model as given in the standard literature usually includes a specification of $p_{\mathrm{up}}$ and $p_{\mathrm{down}} = 1 - p_{\mathrm{up}}$.

Now we are in a position to formulate the following important

**Theorem 2.1:** Let $S$ be some tradable asset whose price process is given by the Binomial model (15). Let $r \geq 0$ denote some constant interest rate. Then every option payoff $H = H(S_0, ..., S_N)$ can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} \tag{16}$$

where we abbreviated

$$V_{k+1}^{\text{up,down}} \;:=\; V_{k+1}\big(S_0, ..., S_k, S_k(1 + \text{ret}_{\text{up,down}})\big) \tag{17}$$

$$S_{k+1}^{\text{up,down}} \;:=\; S_k(1 + \text{ret}_{\text{up,down}}) \tag{18}$$

The portfolio values $V_k$, including the theoretical fair value $V_0$, can be inductively calculated through the formulae

$$V_k \;=\; e^{-r\Delta t} \left\{ p_{\text{rn}}\, V_{k+1}^{\text{up}} \;+\; (1 - p_{\text{rn}})\, V_{k+1}^{\text{down}} \right\} \tag{19}$$

$$V_N \;=\; H$$

with $\Delta t := t_{k+1} - t_k$ which we assume to be constant and with risk neutral probabilities $p_{\text{risk neutral}} =: p_{\text{rn}}$ given by

$$p_{\text{rn}} \;:=\; \frac{e^{r\Delta t} - 1 - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \tag{20}$$

**Remark:** Assume zero interest rates such that $e^{-r\Delta t} = 1$. Then (19) reduces to

$$V_k \;=\; \frac{-\text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}\, V_{k+1}^{\text{up}} \;+\; \frac{\text{ret}_{\text{up}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}\, V_{k+1}^{\text{down}} \tag{21}$$

If we further assume a "symmetric" Binomial model with $\text{ret}_{\text{up}} = +q$ and $\text{ret}_{\text{down}} = -q$ we obtain the simple recursion formula

$$V_k \;=\; \frac{V_{k+1}^{\text{up}} + V_{k+1}^{\text{down}}}{2} \tag{22}$$

Machen Sie sich vielleicht nochmal klar, dass die Formel (19) von oben tatsächlich dieselbe ist wie die Formel (2.14) aus dem FM1-Skript. Im Skript werden risikoneutrale Wahrscheinlichkeiten erst im 3. Kapitel eingeführt, und die analoge Formulierung zu (19) von oben ist im Skript das Korollar 3.1, da die Formel (3.22).

In der nächsten Vorlesung müssen wir uns dann an die Brownsche Bewegung und das Wiener-Maß erinnern, diese Sachen werden wir dann auch gleich in der Finanzmathematik II benutzen müssen.