Hochschule RheinMain
Prof. Dr. D. Lehmann SS 2022

Lösungen zum 4. Übungsblatt
Finanzmathematik II

1.Aufgabe: Wir haben

$$
\begin{aligned}
\mathsf{P}\big[\,\tau_a \in (t, t+dt]\,\big] &= \mathsf{P}\big[\,\tau_a \leq t + dt\,\big] \;-\; \mathsf{P}\big[\,\tau_a \leq t\,\big] \\
&= \mathsf{P}\big[\,\max_{s\in[0,t+dt]} x_s \geq a\,\big] \;-\; \mathsf{P}\big[\,\max_{s\in[0,t]} x_s \geq a\,\big] \\
&= 2\Big\{\mathsf{P}\big[\,x_{t+dt} \geq a\,\big] \;-\; \mathsf{P}\big[\,x_t \geq a\,\big]\Big\}
\end{aligned}
$$

Nun ist

$$
\begin{aligned}
\mathsf{P}\big[\,x_t \geq a\,\big] &= 1 \;-\; \mathsf{P}\big[\,x_t < a\,\big] \\
&= 1 \;-\; \int_{-\infty}^{a} e^{-\frac{x_t^2}{2t}}\,\tfrac{dx_t}{\sqrt{2\pi t}} \\
&= 1 \;-\; \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}}\,\tfrac{dy}{\sqrt{2\pi}} \\
&= 1 \;-\; N\big(\tfrac{a}{\sqrt{t}}\big)
\end{aligned}
$$

und damit

$$
\begin{aligned}
\mathsf{P}\big[\,\tau_a \in (t, t+dt]\,\big] &= -2\Big\{N\big(\tfrac{a}{\sqrt{t+dt}}\big) \;-\; N\big(\tfrac{a}{\sqrt{t}}\big)\Big\} \\
&= -2\tfrac{d}{dt}N\big(\tfrac{a}{\sqrt{t}}\big)\,dt \\
&= -2(-\tfrac{1}{2})\tfrac{a}{t^{3/2}}N'\big(\tfrac{a}{\sqrt{t}}\big)\,dt \\
&= \tfrac{a}{\sqrt{2\pi}\,t^{3/2}}\,e^{-\frac{a^2}{2t}}\,dt\;.
\end{aligned}
$$

2.Aufgabe: Wegen

$$
\begin{aligned}
\mathsf{P}\big[\,x_t \leq a\,\big] &= \int_{\mathbb{R}} \chi(\,x_t \leq a\,)\,p_{t-0}(0, x_t)\,dx_t \\
&= \int_{-\infty}^{a} e^{-\frac{x_t^2}{2t}}\,\tfrac{dx_t}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}}\,\tfrac{dy}{\sqrt{2\pi}} \\
&= N\big(\tfrac{a}{\sqrt{t}}\big)
\end{aligned}
$$

erhalten wir:

a)

$$\lim_{t\to\infty} \mathsf{P}[\,x_t \le a\,] \;=\; \lim_{t\to\infty} N\big(\tfrac{a}{\sqrt{t}}\big) \;=\; N(0) \;=\; 1/2$$

b)

$$\lim_{t\to\infty} \mathsf{P}[\,x_t \le \sigma\sqrt{t}\,] \;=\; N(\sigma)$$

c)

$$\lim_{t\to\infty} \mathsf{P}[\,x_t \le ct\,] \;=\; \lim_{t\to\infty} N(c\sqrt{t}) \;=\; \begin{cases} N(+\infty) = 1 & \text{falls } c > 0 \\ N(-\infty) = 0 & \text{falls } c < 0\,. \end{cases}$$

Wegen $(a > 0)$

$$\begin{aligned}
\mathsf{P}\big[\max_{s\in[0,t]} x_s \le a\big] &= 1 - \mathsf{P}\big[\max_{s\in[0,t]} x_s > a\big] \\
&\overset{\text{Thm.10.5.a}}{=} 1 - 2\mathsf{P}[\,x_t > a\,] \\
&= 1 - 2\big\{1 - \mathsf{P}[\,x_t \le a\,]\big\} \\
&= 2\mathsf{P}[\,x_t \le a\,] - 1 \\
&= 2N\big(\tfrac{a}{\sqrt{t}}\big) - 1
\end{aligned}$$

erhalten wir weiter:

d)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \le a\big] \;=\; \lim_{t\to\infty} \Big[2N\big(\tfrac{a}{\sqrt{t}}\big) - 1\Big] \;=\; 0$$

e)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \le \sigma\sqrt{t}\big] \;=\; 2N(\sigma) - 1$$

f)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \le ct\big] \;=\; \lim_{t\to\infty} \big[2N(c\sqrt{t}) - 1\big] \;\overset{c\ge 0}{=}\; 1$$