

Lösungen 3. Übungsblatt Finanzmathematik II

1. Aufgabe: Wir haben

$$\begin{aligned} \mathbb{P}[\tau_a \in (t, t + dt)] &= \mathbb{P}[\tau_a \leq t + dt] - \mathbb{P}[\tau_a \leq t] \\ &= \mathbb{P}\left[\max_{s \in [0, t+dt]} x_s \geq a\right] - \mathbb{P}\left[\max_{s \in [0, t]} x_s \geq a\right] \\ &= 2\left\{\mathbb{P}[x_{t+dt} \geq a] - \mathbb{P}[x_t \geq a]\right\} \end{aligned}$$

Nun ist

$$\begin{aligned} \mathbb{P}[x_t \geq a] &= 1 - \mathbb{P}[x_t < a] \\ &= 1 - \int_{-\infty}^a e^{-\frac{x_t^2}{2t}} \frac{dx_t}{\sqrt{2\pi t}} \\ &= 1 - \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= 1 - N\left(\frac{a}{\sqrt{t}}\right) \end{aligned}$$

und damit

$$\begin{aligned} \mathbb{P}[\tau_a \in (t, t + dt)] &= -2\left\{N\left(\frac{a}{\sqrt{t+dt}}\right) - N\left(\frac{a}{\sqrt{t}}\right)\right\} \\ &= -2\frac{d}{dt}N\left(\frac{a}{\sqrt{t}}\right) dt \\ &= -2\left(-\frac{1}{2}\right)\frac{a}{t^{3/2}}N'\left(\frac{a}{\sqrt{t}}\right) dt \\ &= \frac{a}{\sqrt{2\pi}t^{3/2}} e^{-\frac{a^2}{2t}} dt . \end{aligned}$$

2. Aufgabe: Wegen

$$\begin{aligned} \mathbb{P}[x_t \leq a] &= \int_{\mathbb{R}} \chi(x_t \leq a) p_{t-0}(0, x_t) dx_t \\ &= \int_{-\infty}^a e^{-\frac{x_t^2}{2t}} \frac{dx_t}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= N\left(\frac{a}{\sqrt{t}}\right) \end{aligned}$$

erhalten wir:

a)

$$\lim_{t \rightarrow \infty} \mathbb{P}[x_t \leq a] = \lim_{t \rightarrow \infty} N\left(\frac{a}{\sqrt{t}}\right) = N(0) = 1/2$$

b)

$$\lim_{t \rightarrow \infty} \mathbb{P}[x_t \leq \sigma\sqrt{t}] = N(\sigma)$$

c)

$$\lim_{t \rightarrow \infty} \mathbb{P}[x_t \leq ct] = \lim_{t \rightarrow \infty} N(c\sqrt{t}) = \begin{cases} N(+\infty) = 1 & \text{falls } c > 0 \\ N(-\infty) = 0 & \text{falls } c < 0. \end{cases}$$

Wegen ($a > 0$)

$$\begin{aligned} \mathbb{P}\left[\max_{s \in [0, t]} x_s \leq a\right] &= 1 - \mathbb{P}\left[\max_{s \in [0, t]} x_s > a\right] \\ &\stackrel{\text{Thm.10.5.a}}{=} 1 - 2\mathbb{P}[x_t > a] \\ &= 1 - 2\{1 - \mathbb{P}[x_t \leq a]\} \\ &= 2\mathbb{P}[x_t \leq a] - 1 \\ &= 2N\left(\frac{a}{\sqrt{t}}\right) - 1 \end{aligned}$$

erhalten wir weiter:

d)

$$\lim_{t \rightarrow \infty} \mathbb{P}\left[\max_{s \in [0, t]} x_s \leq a\right] = \lim_{t \rightarrow \infty} \left[2N\left(\frac{a}{\sqrt{t}}\right) - 1\right] = 0$$

e)

$$\lim_{t \rightarrow \infty} \mathbb{P}\left[\max_{s \in [0, t]} x_s \leq \sigma\sqrt{t}\right] = 2N(\sigma) - 1$$

f)

$$\lim_{t \rightarrow \infty} \mathbb{P}\left[\max_{s \in [0, t]} x_s \leq ct\right] = \lim_{t \rightarrow \infty} \left[2N(c\sqrt{t}) - 1\right] \stackrel{c > 0}{=} 1$$

3.Aufgabe: Mit Teil (b) von Theorem 10.5 erhalten wir

a)

$$\begin{aligned} \mathbb{P}\left[x_T \leq 1 \wedge \max_{t \in [0, T]} x_t \leq 2\right] &= N\left(\frac{1}{\sqrt{1}}\right) + N\left(\frac{2 \times 2 - 1}{\sqrt{1}}\right) - 1 \\ &= N(1) + N(3) - 1 \approx 0.84 \end{aligned}$$

b)

$$\begin{aligned} \mathbb{P}[x_T \leq 1 \wedge \max_{t \in [0, T]} x_t \leq 2] &= N\left(\frac{1}{\sqrt{100}}\right) + N\left(\frac{2 \times 2 - 1}{\sqrt{100}}\right) - 1 \\ &= N(0.1) + N(0.3) - 1 \approx 0.16 \end{aligned}$$

c)

$$\begin{aligned} \mathbb{P}[x_T \geq -3 \wedge \min_{t \in [0, T]} x_t \geq -6] &= \mathbb{P}[-x_T \leq 3 \wedge -\min_{t \in [0, T]} x_t \leq 6] \\ &= \mathbb{P}[-x_T \leq 3 \wedge \max_{t \in [0, T]} \{-x_t\} \leq 6] \\ &= \mathbb{P}[x_T \leq 3 \wedge \max_{t \in [0, T]} \{x_t\} \leq 6] \\ &= N\left(\frac{3}{\sqrt{9}}\right) + N\left(\frac{2 \times 6 - 3}{\sqrt{9}}\right) - 1 \\ &= N(1) + N(3) - 1 \approx 0.84 . \end{aligned}$$