Hochschule RheinMain
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Lösungen 1. Übungsblatt Finanzmathematik II

1.Aufgabe: **a)** Nach Theorem 4.1 haben wir

$$\begin{aligned}
\mathsf{E}[e^{\sigma x_t}] &= \int_{\mathbb{R}} e^{\sigma x_t}\, p_{t-0}(x_0, x_t)\, dx_t \\
&= \int_{\mathbb{R}} e^{\sigma x_t}\, e^{-\frac{x_t^2}{2t}}\, \tfrac{dx_t}{\sqrt{2\pi t}} \\
&= \int_{\mathbb{R}} e^{\sigma x \sqrt{t}}\, e^{-\frac{x^2}{2}}\, \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{\frac{\sigma^2}{2}t} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{t}\,x + \sigma^2 t\right)}\, \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{\frac{\sigma^2}{2}t} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\sigma\sqrt{t})^2}\, \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{\frac{\sigma^2}{2}t} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\, \tfrac{dy}{\sqrt{2\pi}} \\
&= e^{\frac{\sigma^2}{2}t}\,.
\end{aligned}$$

b) Wir bekommen

$$\begin{aligned}
\mathsf{E}[e^{\lambda x_t^2}] &= \int_{\mathbb{R}} e^{\lambda x_t^2}\, p_{t-0}(x_0, x_t)\, dx_t \\
&= \int_{\mathbb{R}} e^{\lambda x_t^2}\, e^{-\frac{x_t^2}{2t}}\, \tfrac{dx_t}{\sqrt{2\pi t}} \\
&= \int_{\mathbb{R}} e^{\lambda x^2 t}\, e^{-\frac{x^2}{2}}\, \tfrac{dx}{\sqrt{2\pi}} \\
&= \int_{\mathbb{R}} e^{-\frac{1}{2}(1-2\lambda t)x^2}\, \tfrac{dx}{\sqrt{2\pi}}
\end{aligned}$$

Das obige Integral existiert nur, wenn der Exponent negativ ist, also wenn der Ausdruck in den runden Klammern positiv ist:

$$\begin{aligned}
\alpha := 1 - 2\lambda t &> 0 \\
\Leftrightarrow \qquad \lambda\, t &< \frac{1}{2}
\end{aligned}$$

In dem Fall erhalten wir:

$$\begin{aligned}\mathsf{E}[e^{\lambda x_t^2}] &= \int_{\mathbb{R}} e^{-\frac{1}{2}\alpha x^2}\,\tfrac{dx}{\sqrt{2\pi}} \\ &= \tfrac{1}{\sqrt{\alpha}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\,\tfrac{dy}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{1-2\lambda t}}\,.\end{aligned}$$

2.Aufgabe: **a)** Der Zeit-t Preis V_t einer Option H mit Laufzeit $T > t$ im Black-Scholes Modell ist gegeben durch (für beliebige Zinsen $r \in \mathbb{R}$)

$$\begin{aligned}V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H\left(S_T\right)\, e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H\left(S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\frac{\sigma^2}{2})(T-t)}\right)\, e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\ &= 100\, e^{-r(T-t)} \int_{\mathbb{R}} \chi\left(S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\frac{\sigma^2}{2})(T-t)} \geq K\right)\, e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}}\end{aligned}$$

wobei die χ-Funktion wie üblich definiert ist durch

$$\chi(A) = \begin{cases} 1 & \text{falls A wahr ist} \\ 0 & \text{falls A falsch ist} \end{cases}$$

Nun ist

$$\begin{aligned} & S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\frac{\sigma^2}{2})(T-t)} \geq K \\ \Leftrightarrow\quad & \sigma\sqrt{T-t}\,x + (r-\tfrac{\sigma^2}{2})(T-t) \geq \log\left[\tfrac{K}{S_t}\right] \\ \Leftrightarrow\quad & \sigma\sqrt{T-t}\,x \geq \log\left[\tfrac{K}{S_t}\right] - (r-\tfrac{\sigma^2}{2})(T-t) \\ \Leftrightarrow\quad & x \geq \frac{\log\left[\frac{K}{S_t}\right] - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ \Leftrightarrow\quad & x \geq -\frac{\log\left[\frac{S_t}{K}\right] + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad =: \quad -d_-\end{aligned}$$

wobei das d_- mit dem d_- aus den Standard Black-Scholes Formeln für Call- und Put-Optionen übereinstimmt. Also:

$$\begin{aligned}V_t &= 100\, e^{-r(T-t)} \int_{\mathbb{R}} \chi\left(S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\frac{\sigma^2}{2})(T-t)} \geq K\right)\, e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\ &= 100\, e^{-r(T-t)} \int_{-d_-}^{\infty} e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\ &= 100\, e^{-r(T-t)} \int_{-\infty}^{d_-} e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \quad = \quad 100\, e^{-r(T-t)}\, N(d_-)\,.\end{aligned}$$

Für Zinsen $r = 0$ erhalten wir also:

$$V_t \;=\; 100\,N(d_-) \tag{1}$$

mit einem $d_- = d_-(S_t, t)$ gegeben durch

$$d_- \;=\; \frac{\log\left[\frac{S_t}{K}\right] \;-\; \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \;=\; \frac{\log\left[\frac{S_t}{K}\right]}{\sigma\sqrt{T-t}} \;-\; \frac{\sigma\sqrt{T-t}}{2} \;=\; d_-(S_t, t)\;. \tag{2}$$

b) Die Black-Scholes PDE mit Zinsen $r = 0$ lautet:

$$\frac{\partial V}{\partial t} \;+\; \frac{\sigma^2}{2}\,S^2\,\frac{\partial^2 V}{\partial S^2} \;=\; 0$$

Mit den Formeln (1) und (2) aus Teil (a) erhalten wir mit $V = V_t = V(S_t, t)$:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= 100\,N'(d_-) \times \frac{\partial d_-}{\partial t} \\
\frac{\partial V}{\partial S} &= 100\,N'(d_-) \times \frac{\partial d_-}{\partial S} \\
\frac{\partial^2 V}{\partial S^2} &= 100\,N''(d_-) \times \left(\frac{\partial d_-}{\partial S}\right)^2 \;+\; 100\,N'(d_-) \times \frac{\partial^2 d_-}{\partial S^2}
\end{aligned}$$

Aus Formel (2) erhalten wir:

$$\begin{aligned}
\frac{\partial d_-}{\partial t} &= \frac{1}{2}\,\frac{\log\left[\frac{S_t}{K}\right]}{\sigma\sqrt{T-t}^{\,3}} \;+\; \frac{\sigma}{4\sqrt{T-t}} \\
\frac{\partial d_-}{\partial S_t} &= \frac{1}{S_t\,\sigma\sqrt{T-t}} \\
\frac{\partial^2 d_-}{\partial S^2} &= -\frac{1}{S_t^2\,\sigma\sqrt{T-t}}
\end{aligned}$$

Weiterhin ist

$$\begin{aligned}
N''(x) &= \frac{d}{dx}\left(\tfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) \\
&= -x \times \left(\tfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) \;=\; -x\,N'(x)
\end{aligned}$$

Damit erhalten wir ($S = S_t$):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} =$$

$$100\, N'(d_-) \times \left\{ \frac{\partial d_-}{\partial t} + \frac{\sigma^2}{2} S_t^2 \left[-d_- \left(\frac{\partial d_-}{\partial S} \right)^2 + \frac{\partial^2 d_-}{\partial S^2} \right] \right\}$$

$$100\, N'(d_-) \times \left\{ \frac{1}{2} \frac{\log\left[\frac{S_t}{K}\right]}{\sigma \sqrt{T-t}^3} + \frac{\sigma}{4\sqrt{T-t}} + \frac{\sigma^2}{2} S_t^2 \left[-\left(\frac{\log\left[\frac{S_t}{K}\right]}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2} \right) \frac{1}{S_t^2\, \sigma^2 (T-t)} - \frac{1}{S_t^2\, \sigma \sqrt{T-t}} \right] \right\}$$

$$100\, N'(d_-) \times \left\{ \frac{1}{2} \frac{\log\left[\frac{S_t}{K}\right]}{\sigma \sqrt{T-t}^3} + \frac{\sigma}{4\sqrt{T-t}} + \frac{1}{2} \left[-\left(\frac{\log\left[\frac{S_t}{K}\right]}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2} \right) \frac{1}{(T-t)} - \frac{\sigma}{\sqrt{T-t}} \right] \right\}$$

$$100\, N'(d_-) \times \left\{ \frac{1}{2} \frac{\log\left[\frac{S_t}{K}\right]}{\sigma \sqrt{T-t}^3} + \frac{\sigma}{4\sqrt{T-t}} - \frac{1}{2} \frac{\log\left[\frac{S_t}{K}\right]}{\sigma \sqrt{T-t}^3} + \frac{\sigma}{4\sqrt{T-t}} - \frac{\sigma}{2\sqrt{T-t}} \right\} = 0\,.$$

c) Das Black-Scholes delta ist gegeben durch

$$\begin{aligned} \delta_t &= \frac{\partial V}{\partial S} \\ &= 100\, N'(d_-) \times \frac{\partial d_-}{\partial S} \\ &= \frac{100}{S_t\, \sigma \sqrt{2\pi(T-t)}} \times e^{-\frac{d_-^2}{2}} \end{aligned}$$

d) Mit Hilfe der Ito-Formel erhalten wir (für Zinsen $r = 0$)

$$\begin{aligned} H(S_T) - V_0 &= V(S_T, T) - V(S_0, 0) \\ &= \int_0^T dV \\ &= \int_0^T \left\{ \frac{\partial V}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} (dS_t)^2 + \frac{\partial V}{\partial t} dt \right\} \\ &= \int_0^T \left\{ \delta_t\, dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 (dx_t)^2 + \frac{\partial V}{\partial t} dt \right\} \\ &= \int_0^T \delta_t\, dS_t + \int_0^T \underbrace{\left\{ \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 + \frac{\partial V}{\partial t} \right\}}_{=0} dt = \int_0^T \delta_t\, dS_t\,. \end{aligned}$$