Hochschule RheinMain SS 2017
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Lösungen 8. Übungsblatt Finanzmathematik II

1.Aufgabe: **a)** Wir haben mit $F(y) = y^2$

$$\begin{aligned}\int_{\mathbb{R}} F\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)e^{-\frac{x^2}{2}}\tfrac{dx}{\sqrt{2\pi}} &= \int_{\mathbb{R}}\big(\sigma_{\mathrm{imp},T}\sqrt{T}\,x\big)^2 e^{-\frac{x^2}{2}}\tfrac{dx}{\sqrt{2\pi}} \\ &= \sigma^2_{\mathrm{imp},T}\,T\int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}}\tfrac{dx}{\sqrt{2\pi}} \\ &= \sigma^2_{\mathrm{imp},T}\,T\times 1 \\ &= \int_0^T \sigma_t^2\,dt\end{aligned}$$

b) Wir bekommen

$$\begin{aligned}\mathsf{E}_W\Big[\big(\textstyle\int_0^T\sigma_t\,dx_t\big)^2\Big] &= \mathsf{E}_W\Big[\textstyle\int_0^T\sigma_t\,dx_t\times\int_0^T\sigma_u\,dx_u\Big] \\ &= \lim_{\Delta t\to 0}\mathsf{E}_W\Big[\sum_{j=1}^{N_T}\sigma_{t_j}\sqrt{\Delta t}\,\phi_j\times\sum_{k=1}^{N_T}\sigma_{t_k}\sqrt{\Delta t}\,\phi_k\Big] \\ &= \lim_{\Delta t\to 0}\sum_{j=1}^{N_T}\sigma_{t_j}\sqrt{\Delta t}\sum_{k=1}^{N_T}\sigma_{t_k}\sqrt{\Delta t}\;\mathsf{E}_W[\,\phi_j\times\phi_k\,] \\ &= \lim_{\Delta t\to 0}\sum_{j,k=1}^{N_T}\sigma_{t_j}\sigma_{t_k}\Delta t\;\mathsf{E}_W[\,\phi_j\phi_k\,]\end{aligned}$$

Wegen

$$dW = \prod_{k=1}^{N_T} e^{-\frac{\phi_k^2}{2}}\tfrac{d\phi_k}{\sqrt{2\pi}}$$

haben wir für $j\neq k$

$$\begin{aligned}\mathsf{E}_W[\,\phi_j\phi_k\,] &= \int_{\mathbb{R}^{N_T}}\phi_j\phi_k\prod_{l=1}^{N_T}e^{-\frac{\phi_l^2}{2}}\tfrac{d\phi_l}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}^2}\phi_j\phi_k\,e^{-\frac{\phi_j^2}{2}}e^{-\frac{\phi_k^2}{2}}\tfrac{d\phi_j}{\sqrt{2\pi}}\tfrac{d\phi_k}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}}\phi_j\,e^{-\frac{\phi_j^2}{2}}\tfrac{d\phi_j}{\sqrt{2\pi}}\times\int_{\mathbb{R}}\phi_k\,e^{-\frac{\phi_k^2}{2}}\tfrac{d\phi_k}{\sqrt{2\pi}} \;=\; 0\end{aligned}$$

und für $j = k$

$$\begin{aligned}
\mathsf{E}_W[\,\phi_j\phi_j\,] &= \int_{\mathbb{R}^{N_T}} \phi_j^2 \prod_{l=1}^{N_T} e^{-\frac{\phi_l^2}{2}} \frac{d\phi_l}{\sqrt{2\pi}} \\
&= \int_{\mathbb{R}} \phi_j^2 \, e^{-\frac{\phi_j^2}{2}} \frac{d\phi_j}{\sqrt{2\pi}} \;=\; 1
\end{aligned}$$

so dass

$$\begin{aligned}
\mathsf{E}_W\Big[\big(\textstyle\int_0^T \sigma_t\,dx_t\big)^2\Big] &= \lim_{\Delta t\to 0} \sum_{j,k=1}^{N_T} \sigma_{t_j}\sigma_{t_k}\Delta t \; \mathsf{E}_W[\,\phi_j\phi_k\,] \\
&= \lim_{\Delta t\to 0} \sum_{j=1}^{N_T} \sigma_{t_j}\sigma_{t_j}\,\Delta t \times 1 \\
&= \int_0^T \sigma_{t_j}^2 \; dt
\end{aligned}$$

c)] Analog zu (b) haben wir für beliebige Funktionen $f, g : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned}
\mathsf{E}_W\Big[\; \textstyle\int_0^T f(t)\,dx_t \times \int_0^T g(t)\,dx_t \;\Big] &= \lim_{\Delta t\to 0} \sum_{j,k=1}^{N_T} f(t_j)\,g(t_k)\,\Delta t \; \mathsf{E}_W[\,\phi_j\phi_k\,] \\
&= \lim_{\Delta t\to 0} \sum_{j=1}^{N_T} f(t_j)\,g(t_j)\,\Delta t \;\times 1 \\
&= \int_0^T f(t)g(t)\,dt \;.
\end{aligned}$$