Hochschule RheinMain
Prof. Dr. D. Lehmann

Lösungen 3. Übungsblatt Finanzmathematik II

1.Aufgabe: Wir haben

$$\begin{aligned} \mathsf{P}\big[\,\tau_a \in (t, t+dt]\,\big] &= \mathsf{P}\big[\,\tau_a \le t+dt\,\big] \;-\; \mathsf{P}\big[\,\tau_a \le t\,\big] \\ &= \mathsf{P}\big[\max_{s\in[0,t+dt]} x_s \ge a\,\big] \;-\; \mathsf{P}\big[\max_{s\in[0,t]} x_s \ge a\,\big] \\ &= 2\Big\{\mathsf{P}\big[\,x_{t+dt} \ge a\,\big] \;-\; \mathsf{P}\big[\,x_t \ge a\,\big]\Big\} \end{aligned}$$

Nun ist

$$\begin{aligned} \mathsf{P}\big[\,x_t \ge a\,\big] &= 1 \;-\; \mathsf{P}\big[\,x_t < a\,\big] \\ &= 1 \;-\; \int_{-\infty}^{a} e^{-\frac{x_t^2}{2t}} \frac{dx_t}{\sqrt{2\pi t}} \\ &= 1 \;-\; \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= 1 \;-\; N\big(\tfrac{a}{\sqrt{t}}\big) \end{aligned}$$

und damit

$$\begin{aligned} \mathsf{P}\big[\,\tau_a \in (t, t+dt]\,\big] &= -2\Big\{N\big(\tfrac{a}{\sqrt{t+dt}}\big) \;-\; N\big(\tfrac{a}{\sqrt{t}}\big)\Big\} \\ &= -2\tfrac{d}{dt} N\big(\tfrac{a}{\sqrt{t}}\big)\, dt \\ &= -2(-\tfrac{1}{2})\tfrac{a}{t^{3/2}} N'\big(\tfrac{a}{\sqrt{t}}\big)\, dt \\ &= \tfrac{a}{\sqrt{2\pi}\, t^{3/2}}\, e^{-\frac{a^2}{2t}}\, dt\;. \end{aligned}$$

2.Aufgabe: Wegen

$$\begin{aligned} \mathsf{P}\big[\,x_t \le a\,\big] &= \int_{\mathbb{R}} \chi(\,x_t \le a\,)\, p_{t-0}(0, x_t)\, dx_t \\ &= \int_{-\infty}^{a} e^{-\frac{x_t^2}{2t}} \frac{dx_t}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= N\big(\tfrac{a}{\sqrt{t}}\big) \end{aligned}$$

erhalten wir:

a)

$$\lim_{t\to\infty} \mathsf{P}[\, x_t \leq a \,] \;=\; \lim_{t\to\infty} N\big(\tfrac{a}{\sqrt{t}}\big) \;=\; N(0) \;=\; 1/2$$

b)

$$\lim_{t\to\infty} \mathsf{P}[\, x_t \leq \sigma\sqrt{t} \,] \;=\; N(\sigma)$$

c)

$$\lim_{t\to\infty} \mathsf{P}[\, x_t \leq ct \,] \;=\; \lim_{t\to\infty} N(c\sqrt{t}) \;=\; \begin{cases} N(+\infty) = 1 & \text{falls } c > 0 \\ N(-\infty) = 0 & \text{falls } c < 0\,. \end{cases}$$

Wegen $(a > 0)$

$$\begin{aligned} \mathsf{P}\big[\max_{s\in[0,t]} x_s \leq a\big] \;&=\; 1 \;-\; \mathsf{P}\big[\max_{s\in[0,t]} x_s > a\big] \\ &\overset{\text{Thm.10.5.a}}{=}\; 1 \;-\; 2\mathsf{P}[\, x_t > a \,] \\ &=\; 1 \;-\; 2\big\{1 - \mathsf{P}[\, x_t \leq a \,]\big\} \\ &=\; 2\mathsf{P}[\, x_t \leq a \,] \;-\; 1 \\ &=\; 2N\big(\tfrac{a}{\sqrt{t}}\big) \;-\; 1 \end{aligned}$$

erhalten wir weiter:

d)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \leq a\big] \;=\; \lim_{t\to\infty}\Big[2N\big(\tfrac{a}{\sqrt{t}}\big) \;-\; 1\Big] \;=\; 0$$

e)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \leq \sigma\sqrt{t}\big] \;=\; 2N(\sigma) \;-\; 1$$

f)

$$\lim_{t\to\infty} \mathsf{P}\big[\max_{s\in[0,t]} x_s \leq ct\big] \;=\; \lim_{t\to\infty}\big[2N(c\sqrt{t}) \;-\; 1\big] \;\overset{c\geq 0}{=}\; 1$$

3.Aufgabe: Mit Teil (b) von Theorem 10.5 erhalten wir

a)

$$\begin{aligned} \mathsf{P}\big[\, x_T \leq 1 \wedge \max_{t\in[0,T]} x_t \leq 2 \,\big] \;&=\; N\big(\tfrac{1}{\sqrt{1}}\big) + N\big(\tfrac{2\times 2-1}{\sqrt{1}}\big) - 1 \\ &=\; N(1) + N(3) - 1 \;\approx\; 0.84 \end{aligned}$$

b)

$$\mathsf{P}\Big[\, x_T \le 1 \;\wedge\; \max_{t\in[0,T]} x_t \le 2 \,\Big] \;=\; N\Big(\tfrac{1}{\sqrt{100}}\Big) + N\Big(\tfrac{2\times 2-1}{\sqrt{100}}\Big) - 1$$
$$=\; N(0.1) + N(0.3) - 1 \;\approx\; 0.16$$

c)

$$\mathsf{P}\Big[\, x_T \ge -3 \;\wedge\; \min_{t\in[0,T]} x_t \ge -6 \,\Big] \;=\; \mathsf{P}\Big[\, -x_T \le 3 \;\wedge\; -\min_{t\in[0,T]} x_t \le 6 \,\Big]$$
$$=\; \mathsf{P}\Big[\, -x_T \le 3 \;\wedge\; \max_{t\in[0,T]} \{-x_t\} \le 6 \,\Big]$$
$$=\; \mathsf{P}\Big[\, x_T \le 3 \;\wedge\; \max_{t\in[0,T]} \{x_t\} \le 6 \,\Big]$$
$$=\; N\Big(\tfrac{3}{\sqrt{9}}\Big) + N\Big(\tfrac{2\times 6-3}{\sqrt{9}}\Big) - 1$$
$$=\; N(1) + N(3) - 1 \;\approx\; 0.84\,.$$